Non-equilibrium physics WS 20/21 – Exercise Sheet 9:

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1 Discussion:

i) What is the Knudsen number Kn? How does a Boltzmann gas behave in the limits $Kn \ll 1$ and $Kn \gg 1$?

- The Knudsen number is defined as the ratio of the mean-free time $\tau_{\rm mfp}$ to the timescale over which the system responds to gradients and external forces τ_s

$$\mathrm{Kn} \equiv \frac{\tau_{\rm mfp}}{\tau_s} = \frac{l_{\rm mfp}}{L} \,. \tag{1}$$

Kn characterizes the frequency of collisions over the timescale of variation of macroscopic properties/gradients.

- In the limit $Kn \ll 1 \Rightarrow \tau_{\rm mfp} \ll \tau_s$, Several collisions occur during the response to the gradients and external forces leading to local equilibrium distribution with local m.

- In the limit $Kn \gg 1 \Rightarrow \tau_{\rm mfp} \gg \tau_s$, the system is essentially free streaming up to corrections due to rare collision events.

2 In-class problems:

2.1 Ideal hydrodynamics

Consider a monoatomic Boltzmann gas for which we derived the following balance equations for the mass density $\rho(t,\vec{r}) = m\int_{\vec{p}} f(t,\vec{r},\vec{p})$, the velocity field $\rho(t,\vec{r})\vec{v}(t,\vec{r}) = \int_{\vec{p}} \vec{p} f(t,\vec{r},\vec{p})$ and the internal energy density $e(t,\vec{r}) = \int_{\vec{p}} \frac{[\vec{p}-m\vec{v}(t,\vec{r})]^2}{2m} f(t,\vec{r},\vec{p})$

$$\begin{aligned}
\frac{\partial}{\partial t}\rho(t,\vec{r}) + \vec{\nabla}\Big(\rho(t,\vec{r})\vec{v}(t,\vec{r})\Big) &= 0\,, \\
\frac{\partial}{\partial t}\rho(t,\vec{r})v^i(t,\vec{r}) + \frac{\partial}{\partial r_j}\Big(\rho(t,\vec{r})\vec{v}^i(t,\vec{r})\vec{v}^j(t,\vec{r}) + \Pi^{ij}(t,\vec{r})\Big) &= \frac{\rho(t,\vec{r})}{m}F^i(t,\vec{r})\,, \\
\frac{\partial}{\partial t}e(t,\vec{r}) + \vec{\nabla}\Big(e(t,\vec{r})\vec{v}(t,\vec{r}) + \vec{J}_U(t,\vec{r})\Big) &= -\Pi^{ij}(t,\vec{r})\frac{\partial v^i}{\partial r_j}(t,\vec{r})\,,
\end{aligned}$$

where the $\Pi^{ij}(t,\vec{r})$ denotes the stress tensor

$$\Pi^{ij}(t,\vec{r}) = \rho(t,\vec{r})\left\langle \frac{[\vec{p}^i - m\vec{v}^i(t,\vec{r})]}{m}\frac{[\vec{p}^j - m\vec{v}^j(t,\vec{r})]}{m}\right\rangle_{\vec{p}} = \int_{\vec{p}} \frac{[\vec{p}^i - m\vec{v}^i(t,\vec{r})][\vec{p}^j - m\vec{v}^j(t,\vec{r})]}{m} f(t,\vec{r},\vec{p})$$

and $\vec{J}_U(t,\vec{r})$ is the energy flux in the local rest frame

$$\vec{J}_U(t,\vec{r}) = \frac{1}{2}\rho(t,\vec{r})\left\langle \left(\vec{p}/m - \vec{v}(t,\vec{r})\right)^2 [\vec{p}/m - \vec{v}(t,\vec{r})]\right\rangle_{\vec{p}} = \int_{\vec{p}} \frac{\left(\vec{p} - m\vec{v}(t,\vec{r})\right)^2}{2m}[\vec{p}/m - \vec{v}(t,\vec{r})]f(t,\vec{r},\vec{p})$$

i) Calculate the stress-tensor $\Pi^{ij,(0)}(t,\vec{r})$ and energy flux $\vec{J}_U^{(0)}(t,\vec{r})$ for a Boltzmann gas in local thermal equilibrium $f(t,\vec{r},\vec{p}) = f^{(0)}(T(t,\vec{r}),n(t,\vec{r}),\vec{v}(t,\vec{r}),\vec{p})$.

- The local thermal equilibrium distribution is given by

$$f^{(0)}(T(t,\vec{r}),n(t,\vec{r}),\vec{v}(t,\vec{r}),\vec{p}) = n(t,\vec{r})\left(\frac{2\pi\hbar^2}{mk_BT(t,\vec{r})}\right)^{3/2} \exp\left[-\frac{\left(\vec{p}-m\vec{v}(t,\vec{r})\right)^2}{2mk_BT(t,\vec{r})}\right] . \tag{2}$$

- The energy flux $\vec{J}_U^{(0)}(t,\vec{r})$ is written

$$\begin{aligned}\vec{J}_U^{(0)}(t,\vec{r}) &= \int \frac{d^3\vec{p}}{(2\pi\hbar)^3}\frac{\left(\vec{p}-m\vec{v}(t,\vec{r})\right)^2}{2m}[\vec{p}/m - \vec{v}(t,\vec{r})]\ f^{(0)}(T(t,\vec{r}),n(t,\vec{r}),\vec{v}(t,\vec{r}),\vec{p}) \ , &(3)\\ &= 0 \ . &(4)\end{aligned}$$

- The stress tensor is written

$$\begin{aligned}\Pi^{ij,(0)}(t,\vec{r}) &= \int_{\vec{p}} \frac{[\vec{p}^i - m\vec{v}^i(t,\vec{r})][\vec{p}^j - m\vec{v}^j(t,\vec{r})]}{m} f^{(0)}(T(t,\vec{r}),n(t,\vec{r}),\vec{v}(t,\vec{r}),\vec{p}) \ , &(5)\\ &= n(t,\vec{r})\left(\frac{2\pi}{mk_BT(t,\vec{r})}\right)^{3/2}\int\frac{d^3\vec{p}}{(2\pi)^3}\frac{\vec{p}^i\vec{p}^j}{m}\exp\left[-\frac{\sum_k \vec{p}_k^2}{2mk_BT(t,\vec{r})}\right] \ , &(6)\\ &= n(t,\vec{r})\left(\frac{2\pi}{mk_BT(t,\vec{r})}\right)^{3/2}\frac{\pi\sqrt{\pi}}{(2\pi)^3}\frac{\delta^{ij}}{2m}(2mk_BT(t,\vec{r}))^{5/2} \ , &(7)\\ &= n(t,\vec{r})k_BT(t,\vec{r})\delta^{ij} = P(t,\vec{r})\delta^{ij} \ . &(8)\end{aligned}$$

ii) Show that for $\Pi^{ij} = \Pi^{ij,(0)}$ and $\vec{J}_U = \vec{J}_U^{(0)}$ the balance equation reduce to the equations of motion for an ideal fluid

- We write the balance equations using prior results

$$\begin{aligned}\frac{\partial}{\partial t}\rho(t,\vec{r}) &= -\vec{\nabla}\left(\rho(t,\vec{r})\vec{v}(t,\vec{r})\right) \ , &(9)\\ \rho\frac{\partial}{\partial t}v^i + v^i\frac{\partial}{\partial t}\rho + \vec{v}^i\vec{\nabla}_j\rho\vec{v}^j + \rho\vec{v}^j\vec{\nabla}_j\vec{v}^i &= \frac{\rho(t,\vec{r})}{m}F^i(t,\vec{r}) - \frac{\partial}{\partial x^i}P \ , &(10)\\ \rho\left[\frac{\partial\vec{v}}{\partial t} + (\vec{v}\cdot\vec{\nabla})\vec{v}\right] &= \frac{\rho(t,\vec{r})}{m}\vec{F}(t,\vec{r}) - \vec{\nabla}P \ , &(11)\\ \frac{\partial}{\partial t}e(t,\vec{r}) + \vec{\nabla}\left(e(t,\vec{r})\vec{v}(t,\vec{r})\right) &= -P(t,\vec{r})\vec{\nabla}v^i(t,\vec{r}) \ , &(12)\end{aligned}$$

which recover the equations of motion of an ideal fluid (i.e. Navier-Stokes in absence of viscosity).

3 Homework problems:

3.1 Non-equilibrium corrections to ideal hydrodynamics

Based on the Chapman-Enskog expansion the first non-trivial corrections $f^{(1)}(t,\vec{r},\vec{p})$ to the local equilibrium distribution $f^{(0)}(t,\vec{r},\vec{p}) = n(t,\vec{r})\left(\frac{2\pi\hbar^2}{mk_BT(t,\vec{r})}\right)^{3/2}\exp\left(-\frac{[\vec{p}-m\vec{v}(t,\vec{r})]^2}{2mk_BT(t,\vec{r})}\right)$ are determined according to

$$\left(\frac{\partial}{\partial t} + \frac{\vec{p}}{m}\vec{\nabla}_{\vec{r}} + \vec{F}(t,\vec{r})\vec{\nabla}_{\vec{p}}\right) f^{(0)}(t,\vec{r},\vec{p}) = \delta C[f^{(0)}, f^{(1)}](t,\vec{r},\vec{p}) \;, \tag{13}$$

where $\delta C[f^{(0)}, f^{(1)}]$ denotes the linearized collision operator.

i) Show that in the relaxation time approximation the non-equilibrium correction $f^{(1)}(t,\vec{r},\vec{p})$ is given by

$$\begin{aligned} f^{(1)}(t,\vec{r},\vec{p}) &= -\tau_R f^0(t,\vec{r},\vec{p})\left[\frac{1}{\rho(t,\vec{r})}\mathcal{D}\rho(t,\vec{r}) + \left(\frac{m\vec{u}_{\vec{p}}^2}{2k_BT(t,\vec{r})} - \frac{3}{2}\right)\frac{1}{T(t,\vec{r})}\mathcal{D}T(t,\vec{r})\right. \\ &\qquad \left. + \frac{m\vec{u}_{\vec{p}}^i}{k_BT(t,\vec{r})}\mathcal{D}\vec{v}^i(t,\vec{r}) - \frac{\vec{u}_{\vec{p}}}{k_BT(t,\vec{r})}\vec{F}(t,\vec{r})\right] \end{aligned}$$

where $\mathcal{D} = \frac{\partial}{\partial t} + \frac{\vec{p}}{m}\vec{\nabla}_{\vec{r}}$ and we denote $\vec{u}_{\vec{p}} = \vec{p}/m - \vec{v}(t,\vec{r})$.

- In the relaxation time approximation the collision integral is given by

$$\frac{f^{(1)}(t,\vec{r},\vec{p})}{\tau_R} = -\left(\frac{\partial}{\partial t} + \frac{\vec{p}}{m}\vec{\nabla}_{\vec{r}} + \vec{F}(t,\vec{r})\vec{\nabla}_{\vec{p}}\right) f^{(0)}(t,\vec{r},\vec{p}) \;, \tag{14}$$

$$\begin{aligned} f^{(1)}(t,\vec{r},\vec{p}) &= -\tau_R\left(\mathcal{D}n(t,\vec{r})\frac{\partial}{\partial n(t,\vec{r})} + \mathcal{D}T(t,\vec{r})\frac{\partial}{\partial T(t,\vec{r})} + \mathcal{D}\vec{v}^i(t,\vec{r})\frac{\partial}{\partial \vec{v}^i(t,\vec{r})}\right. \\ &\qquad \left. - \frac{\vec{u}_{\vec{p}}}{k_BT(t,\vec{r})}\vec{F}(t,\vec{r})\right) f^{(0)}(t,\vec{r},\vec{p}) \;, \end{aligned} \tag{15}$$

- We have

$$\frac{\partial}{\partial n(t,\vec{r})} f^{(0)}(t,\vec{r},\vec{p}) = m\frac{f^{(0)}(t,\vec{r},\vec{p})}{\rho(t,\vec{r})} \;, \tag{16}$$

$$\frac{\partial}{\partial T(t,\vec{r})} f^{(0)}(t,\vec{r},\vec{p}) = \left(\frac{m\vec{u}_{\vec{p}}^2}{2k_BT(t,\vec{r})} - \frac{3}{2}\right)\frac{1}{T(t,\vec{r})} f^{(0)}(t,\vec{r},\vec{p}) \;, \tag{17}$$

$$\frac{\partial}{\partial \vec{v}^i(t,\vec{r})} f^{(0)}(t,\vec{r},\vec{p}) = \frac{m\vec{u}_{\vec{p}}^i}{k_BT(t,\vec{r})} f^{(0)}(t,\vec{r},\vec{p}) \;. \tag{18}$$

$$\begin{aligned} f^{(1)}(t,\vec{r},\vec{p}) &= -\tau_R f^0(t,\vec{r},\vec{p})\left[\frac{1}{\rho(t,\vec{r})}\mathcal{D}\rho(t,\vec{r}) + \left(\frac{m\vec{u}_{\vec{p}}^2}{2k_BT(t,\vec{r})} - \frac{3}{2}\right)\frac{1}{T(t,\vec{r})}\mathcal{D}T(t,\vec{r})\right. \\ &\qquad \left. + \frac{m\vec{u}_{\vec{p}}^i}{k_BT(t,\vec{r})}\mathcal{D}\vec{v}^i(t,\vec{r}) - \frac{\vec{u}_{\vec{p}}}{k_BT(t,\vec{r})}\vec{F}(t,\vec{r})\right] \;. \end{aligned} \tag{19}$$

ii) Exploit the equations of motions of ideal hydrodynamics

$$\begin{aligned}
\left(\frac{\partial}{\partial t}+\vec{v}(t,\vec{r})\vec{\nabla}\right)\rho(t,\vec{r}) &= -\rho(t,\vec{r})\ \vec{\nabla}\vec{v}(t,\vec{r})\ , \\
\frac{\partial}{\partial t}\vec{v}^i(t,\vec{r})+\vec{v}^j(t,\vec{r})\frac{\partial}{\partial r_j}\vec{v}^i(t,\vec{r}) &= -\frac{1}{\rho(t,\vec{r})}\frac{\partial}{\partial r_i}P(t,\vec{r})+\frac{1}{m}\vec{F}^i(t,\vec{r})\ , \\
\frac{\partial}{\partial t}e(t,\vec{r})+\vec{\nabla}\Big(e(t,\vec{r})\vec{v}(t,\vec{r})\Big) &= -P(t,\vec{r})\ \vec{\nabla}\vec{v}(t,\vec{r})\ ,
\end{aligned}$$

to show that the expression for $f^{(1)}(t,\vec{r},\vec{p})$ can be compactly expressed as

$$\begin{aligned}
f^{(1)}(t,\vec{r},\vec{p}) &= -\tau_R f^0(t,\vec{r},\vec{p})\left[\left(\frac{m\vec{u}_{\vec{p}}^2}{2k_BT(t,\vec{r})}-\frac{5}{2}\right)\ \vec{u}_{\vec{p}}\frac{\vec{\nabla}T(t,\vec{r})}{T(t,\vec{r})}\right. \\
&\qquad \left.+\frac{m}{2k_BT(t,\vec{r})}\left(\frac{\partial v_i}{\partial r_j}(t,\vec{r})+\frac{\partial v_j}{\partial r_i}(t,\vec{r})\right)\left(\vec{u}_{\vec{p}}^i\vec{u}_{\vec{p}}^j-\frac{1}{3}\delta^{ij}\vec{u}_{\vec{p}}^2\right)\right]
\end{aligned}$$

(Hint: Express the energy density and pressure in terms of the temperatures and densities, by using the equations of state $e(t,\vec{r})=\frac{3}{2}n(t,\vec{r})k_BT(t,\vec{r})$ and $P(t,\vec{r})=n(t,\vec{r})k_BT(t,\vec{r})$ to derive the evolution equation for the temperature $T(t,\vec{r})$

$$\left(\frac{\partial}{\partial t}+\vec{v}(t,\vec{r})\vec{\nabla}\right)T(t,\vec{r}) = -\frac{2}{3}T(t,\vec{r})\ \vec{\nabla}\vec{v}(t,\vec{r})\ , \tag{20}$$

Eliminate all time derivatives in the expression for $f^{(1)}$ in favor or spatial derivatives using the ideal equations of motion.)

- The equations of motion of ideal hydrodynamics can be written as

$$\begin{aligned}
\mathcal{D}\rho(t,\vec{r}) &= -\rho(t,\vec{r})\ \vec{\nabla}\vec{v}(t,\vec{r})\ , &(21)\\
\mathcal{D}\vec{v}^i(t,\vec{r}) &= -\frac{1}{\rho(t,\vec{r})}\frac{\partial}{\partial r_i}P(t,\vec{r})+\frac{1}{m}\vec{F}^i(t,\vec{r})\ , &(22)\\
\mathcal{D}\vec{v}^i(t,\vec{r}) &= -\frac{1}{m\rho(t,\vec{r})}\vec{\nabla}^i\rho(t,\vec{r})k_BT(t,\vec{r})+\frac{1}{m}\vec{F}^i(t,\vec{r})\ , &(23)\\
\mathcal{D}\vec{v}^i(t,\vec{r}) &= -\frac{k_B}{m}\vec{\nabla}^iT(t,\vec{r})-\frac{k_BT(t,\vec{r})}{m\rho(t,\vec{r})}\vec{\nabla}^i\rho(t,\vec{r})+\frac{1}{m}\vec{F}^i(t,\vec{r})\ , &\\
\mathcal{D}e(t,\vec{r}) &= -\Big(P(t,\vec{r})+e(t,\vec{r})\Big)\ \vec{\nabla}\vec{v}(t,\vec{r})\ , &(24)\\
\rho(t,\vec{r})\mathcal{D}T(t,\vec{r})+T(t,\vec{r})\mathcal{D}\rho(t,\vec{r}) &= -\frac{5}{3}\rho(t,\vec{r})T(t,\vec{r})\ \vec{\nabla}\vec{v}(t,\vec{r})\ , &(25)\\
\mathcal{D}T(t,\vec{r}) &= -\frac{2}{3}T(t,\vec{r})\ \vec{\nabla}\vec{v}(t,\vec{r})\ . &(26)
\end{aligned}$$

Using this EOM we can write

$$\begin{aligned}
f^{(1)}(t,\vec{r},\vec{p}) &= -\tau_R f^0(t,\vec{r},\vec{p})\left[-\vec{\nabla}\vec{v}(t,\vec{r})-\left(\frac{m\vec{u}_{\vec{p}}^2}{3k_BT(t,\vec{r})}-1\right)\vec{\nabla}\vec{v}(t,\vec{r})\right. \\
&\qquad \left.-\frac{m\vec{u}_{\vec{p}}^i}{k_BT(t,\vec{r})}\left(\frac{k_B}{m}\vec{\nabla}^iT(t,\vec{r})+\frac{k_BT(t,\vec{r})}{m\rho(t,\vec{r})}\vec{\nabla}^i\rho(t,\vec{r})\right)\right]\ .
\end{aligned} \tag{27}$$

$$\begin{aligned}
f^{(1)}(t,\vec{r},\vec{p}) &= -\tau_R f^0(t,\vec{r},\vec{p})\left[\left(\frac{m\vec{u}_{\vec{p}}^2}{2k_BT(t,\vec{r})}-\frac{5}{2}\right)\ \vec{u}_{\vec{p}}\frac{\vec{\nabla}T(t,\vec{r})}{T(t,\vec{r})}\right. \\
&\qquad \left.+\frac{m}{2k_BT(t,\vec{r})}\left(\frac{\partial v_i}{\partial r_j}(t,\vec{r})+\frac{\partial v_j}{\partial r_i}(t,\vec{r})\right)\left(\vec{u}_{\vec{p}}^i\vec{u}_{\vec{p}}^j-\frac{1}{3}\delta^{ij}\vec{u}_{\vec{p}}^2\right)\right]
\end{aligned}$$

iii) Verify explicitly that $f^{(1)}(t,\vec{r},\vec{p})$ does not contribute to the quantities $\rho(t,\vec{r})$, $v(t,\vec{r})$ and $e(t,\vec{r})$. $\Big($Hint: Change integration variables to $d^3\vec{u}_{\vec{p}}$ and consider the symmetries of the integrand$\Big)$

- The corrections due to $f^{(1)}(t,\vec{r},\vec{p})$ are written (using the change of variable $\vec{p}\to\vec{u}_{\vec{p}}$ and $d^3\vec{p}\to m^3 d^3\vec{u}_{\vec{p}}$; and we cancel all odd function of $\vec{u}_{\vec{p}}$)

$$\begin{aligned}
\delta\rho(t,\vec{r}) &= m^4\int\frac{d^3\vec{u}_{\vec{p}}}{(2\pi\hbar)^3}\ f^{(1)}(t,\vec{r},\vec{p})\ , &(28)\\
&= m^4\int\frac{d^3\vec{u}_{\vec{p}}}{(2\pi\hbar)^3}\ -\tau_R f^0(t,\vec{r},\vec{p})\left[\frac{m}{k_BT(t,\vec{r})}\frac{\partial v_i}{\partial r_j}(t,\vec{r})\left(\delta^{ij}\vec{u}_{\vec{p}}^{i2}-\frac{1}{3}\delta^{ij}\vec{u}_{\vec{p}}^2\right)\right]\ , &(29)\\
&= 0\ . &(30)
\end{aligned}$$

which cancels as the integral involving $\frac{1}{3}\delta^{ij}\vec{u}_{\vec{p}}^2$ is equivalent to the one involving $\delta^{ij}\vec{u}_{\vec{p}}^{i2}$ due to the symmetry between x,y,z directions.

$$\begin{aligned}
\delta\vec{v}(t,\vec{r}) &= m^3\int\frac{d^3\vec{u}_{\vec{p}}}{(2\pi\hbar)^3}\ \frac{\vec{p}}{m}f^{(1)}(t,\vec{r},\vec{p})\ , &(31)\\
&= m^3\int\frac{d^3\vec{u}_{\vec{p}}}{(2\pi\hbar)^3}\ \left(\vec{u}_{\vec{p}}+\vec{v}(t,\vec{r})\right)f^1(t,\vec{r},\vec{p})\ , &(32)\\
&= m^3\int\frac{d^3\vec{u}_{\vec{p}}}{(2\pi\hbar)^3}\ \vec{u}_{\vec{p}}f^1(t,\vec{r},\vec{p})+\frac{\delta\rho(t,\vec{r})}{m}\vec{v}(t,\vec{r})\ , &(33)\\
&= -\tau_R m^3\int\frac{d^3\vec{u}_{\vec{p}}}{(2\pi\hbar)^3}\ \vec{u}_{\vec{p}}f^0(t,\vec{r},\vec{p})\left[\left(\frac{m\vec{u}_{\vec{p}}^2}{2k_BT(t,\vec{r})}-\frac{5}{2}\right)\ \vec{u}_{\vec{p}}\frac{\vec{\nabla}T(t,\vec{r})}{T(t,\vec{r})}\right.\\
&\qquad\left.+\frac{m}{2k_BT(t,\vec{r})}\left(\frac{\partial v_i}{\partial r_j}(t,\vec{r})+\frac{\partial v_j}{\partial r_i}(t,\vec{r})\right)\vec{u}_{\vec{p}}^i\vec{u}_{\vec{p}}^j\right]\\
&= 0\ . &(34)
\end{aligned}$$

$$\begin{aligned}
\delta e(t,\vec{r}) &= m^4\int\frac{d^3\vec{u}_{\vec{p}}}{(2\pi\hbar)^3}\ \vec{u}_{\vec{p}}^2 f^{(1)}(t,\vec{r},\vec{p})\ , &(35)\\
&= -\tau_R m^4\int\frac{d^3\vec{u}_{\vec{p}}}{(2\pi\hbar)^3}\ \vec{u}_{\vec{p}}^2 f^0(t,\vec{r},\vec{p})\left[\frac{m}{k_BT(t,\vec{r})}\frac{\partial v_i}{\partial r_j}(t,\vec{r})\left(\vec{u}_{\vec{p}}^i\vec{u}_{\vec{p}}^j\delta^{ij}-\frac{1}{3}\delta^{ij}\vec{u}_{\vec{p}}^2\right)\right]\ , &(36)\\
&= 0\ . &(37)
\end{aligned}$$

iv) Calculate the heat conductivity κ in the constitutive relation $\vec{J}_U=-\kappa\vec{\nabla}T(t,\vec{r})$
$\left(\text{Hint: } \int_{\vec{p}}\frac{(m\vec{u}_{\vec{p}})^2}{2m}(m\vec{u}_{\vec{p}})^2\left(\frac{(m\vec{u}_{\vec{p}})^2}{2mk_BT}-\frac{5}{2}\right)f^{(0)}=\frac{15}{2}mn(k_BT)^2\ \right)$

- The internal energy flux is given by

$$
\begin{aligned}
\vec{J}^i_U &= m^3 \int_{\vec{p}} \vec{u}^i_{\vec{p}} \vec{u}^2_{\vec{p}} \big(f^{(0)} + f^{(1)}\big) \,, && (38)\\
&= m^3 \int_{\vec{p}} \vec{u}^i_{\vec{p}} \vec{u}^2_{\vec{p}} f^{(1)} \,, && (39)\\
&= -\tau_R m^3 \int_{\vec{p}} \vec{u}^i_{\vec{p}} \vec{u}^2_{\vec{p}} f^0(t,\vec{r},\vec{p}) \left[\left(\frac{m\vec{u}^2_{\vec{p}}}{2k_B T(t,\vec{r})} - \frac{5}{2} \right) \vec{u}^j_{\vec{p}} \frac{\vec{\nabla}^j T(t,\vec{r})}{T(t,\vec{r})} \right. \\
&\qquad \left. + \frac{m}{2k_B T(t,\vec{r})} \left(\frac{\partial v_l}{\partial r_k}(t,\vec{r}) + \frac{\partial v_l}{\partial r_k}(t,\vec{r}) \right) \left(\vec{u}^l_{\vec{p}} \vec{u}^k_{\vec{p}} - \frac{1}{3}\delta^{lk} \vec{u}^2_{\vec{p}} \right) \right] \,, && (40)\\
&= -\tau_R \frac{\delta^{ij}}{3} \frac{\vec{\nabla}^j T(t,\vec{r})}{T(t,\vec{r})} \int_{\vec{p}} \frac{(m\vec{u}_{\vec{p}})^2}{2m} (m\vec{u}_{\vec{p}})^2 f^0(t,\vec{r},\vec{p}) \left(\frac{m\vec{u}^2_{\vec{p}}}{2k_B T(t,\vec{r})} - \frac{5}{2} \right) \,, && (41)\\
&= -\frac{5}{2} \tau_R m n k_B^2 T \vec{\nabla} T(t,\vec{r}) \,. && (42)
\end{aligned}
$$

- The heat conductivity is written

$$\kappa = \frac{5}{2} \tau_R m n k_B^2 T \,. \qquad (43)$$