Non-equilibrium physics WS 20/21 – Exercise Sheet 8:

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1 Discussion:

i) Discuss the definition, physical meaning and properties of entropy in classical and quantum statistical system, and in the context of the Boltzmann equation.

- Entropy characterizes the missing information content of a statistical system (larger entropy less information). From information theory we define Shannon entropy

$$S_{\text{Shannon}} = -k \sum_n p_n \log p_n \;, \tag{1}$$

this quantity is positive semi-definite $S_{\text{Shannon}} \geq 0$, where $S_{\text{Shannon}} = 0$ when the state is localized $p_m = \delta_{nm}$.

- In a quantum system the density operator

$$\hat{\rho} = \sum_i p_i |\phi_i\rangle\langle\phi_i| \tag{2}$$

describes the probability for the system to be in a state $|\phi_i\rangle$, and we define the Von Neumann entropy

$$S_q(\hat{\rho}) = -k_B \text{Tr}(\hat{\rho} \log \hat{\rho}) \;, \tag{3}$$

which reduces to the Shannon entropy

$$S_q(\hat{\rho}) = -k_B \sum_n p_i \log p_i \;, \tag{4}$$

with $S_q \geq 0$, and $S_q = 0$ for pure state.

- Entropy is conserved under Hamiltonian time evolution in classical and quantum systems.

- In a classical system described by the probability density $f_N(t, \{r_i\}, \{p_i\})$. We define the entropy

$$S_{\text{cl}} = -k_B \int d^{6N}\mathcal{V} f_N(t, \{r_i\}, \{p_i\}) \log \Big(f_N(t, \{r_i\}, \{p_i\}) \Big) \;. \tag{5}$$

Since $f_N(t, \{r_i\}, \{p_i\})$ is a probability density, f_N is not bound by one it can be larger ($f_N > 1$). The logarithm is then not always negative, one can find $S_{\text{cl}} < 0$. We even find $S_{\text{cl}} \to -\infty$ when all coordinates and momentum are known exactly.

- In the context of the Boltzmann equation one defines the Boltzmann entropy at the level of the single particle distribution

$$S_{\rm B} = -k_B \int d^6\mathcal{V} f_1(t,\{r_i\},\{p_i\}) \log\Big(f_1(t,\{r_i\},\{p_i\})\Big) . \tag{6}$$

Due to the Boltzmann H-theorem $dS_B/dt \geq 0$, the Boltzmann entropy in the Boltzmann equation is monotonically increasing as a function of time.

i) What is the form of the stationary solution of the Boltzmann equation for a homogenous system in the absence of external forces? What is the difference between the concepts of *balance* and *detailed balance*? What is the *relaxation time approximation*?

- The stationary solution of the Boltzmann equation for a homogenous system in the absence of external forces, as derived in the lecture, is given by Maxwell-Boltzmann distribution

$$f_{\rm eq}(\vec{r},\vec{p}) = n\left(\frac{2\pi\hbar^2}{mk_BT}\right)^{3/2} e^{-\frac{\vec{p}^2}{2mk_BT}} \tag{7}$$

- These concepts appear when looking for the zeros of the collision integral. Either the integral sum is vanishing, leading to the *balance* effect where the net effect of all possible interactions balance each other.

$$\int_{\vec{p}_2\vec{p}_3\vec{p}_4} \omega(\vec{p}_1\vec{p}_2 \to \vec{p}_3\vec{p}_4)\,[f_3f_4 - f_1f_2] = 0 . \tag{8}$$

- Or *Detailed balance* when the integrand itself is zero, leading to a stationary solution due to the fact that at equilibrium the net effect of each process is balanced by its inverse process.

$$\omega(\vec{p}_1\vec{p}_2 \to \vec{p}_3\vec{p}_4)\,[f_3f_4 - f_1f_2] = 0 . \tag{9}$$

- The relaxation time approximates the typical duration for reaching local equilibrium. A system near equlibrium will follow an exponential relaxation towards equilibrium and we write the collision integral as follows

$$C[f^{(0)}, f^{(1)}] = -\frac{f^{(1)}(t,\vec{r},\vec{p}) - f^{(0)}(\vec{r},\vec{p})}{\tau_r} \tag{10}$$

2 In-class problems:

2.1 Global equilibrium in the presence of a scalar potential

Consider a dilute gas of particles, whose dynamics is described by the Boltzmann equation, in the presence of an external force $\vec{F}(\vec{r}) = -\vec{\nabla}_{\vec{r}}V(\vec{r})$ derived from a scalar potential $V(\vec{r})$

i) Show that the global equilibrium solution is of the form $f_{\rm eq}(\vec{r},\vec{p}) = n(\vec{r})\left(\frac{2\pi\hbar^2}{mk_BT}\right)^{3/2} e^{-\frac{\vec{p}^2}{2mk_BT}}$ and determine the spatial profile of density $n(\vec{r})$

- The Boltzmann equation is written

$$\left(\partial_t + \frac{\vec{p}}{m}\cdot\vec{\nabla}_r - \vec{\nabla}_{\vec{r}}V(\vec{r})\cdot\vec{\nabla}_p\right) f_{\rm eq}(t,\vec{r},\vec{p}) \;=\; C[f_{\rm eq}](t,\vec{r},\vec{p})\,, \tag{11}$$

$$\frac{\vec{p}}{m}\cdot\vec{\nabla}_r n(\vec{r}) + n(\vec{r})\frac{\vec{p}}{mk_BT}\cdot\vec{\nabla}_{\vec{r}}V(\vec{r}) \;=\; 0\,, \tag{12}$$

$$\tag{13}$$

where the collision term vanishes due to energy conservation (same as in Sheet 7). For arbitrary $\vec{p}$, the density must solve the following PDE

$$\vec{\nabla}_r n(\vec{r}) + n(\vec{r})\frac{\vec{\nabla}_{\vec{r}}V(\vec{r})}{k_BT} \;=\; 0\,, \tag{14}$$

$$n(\vec{r}) \;=\; e^{-\frac{V(\vec{r})-V(\vec{r}_0)}{k_BT}}n(\vec{r}_0)\,. \tag{15}$$

3 Homework problems:

3.1 Boltzmann gas in a harmonic trap

Consider a dilute gas of particles described by the Boltzmann equation

$$\left(\frac{\partial}{\partial t} + \frac{\vec{p}}{m}\vec{\nabla}_{\vec{r}} + \vec{F}(\vec{r})\vec{\nabla}_{\vec{p}}\right) f(t,\vec{r},\vec{p}) = C[f](t,\vec{r},\vec{p})\,, \tag{16}$$

in the presence of an external force $\vec{F}(\vec{r}) = -\vec{\nabla}_{\vec{r}}V(\vec{r})$ derived from a harmonic potential

$$V(\vec{r}) = \frac{1}{2}m\omega^2\vec{r}^2.$$

i) Determine the global equilibrium solution $f_{\rm eq}(\vec{r},\vec{p})$ for this system.

- Following the In-class problem we have

$$n(\vec{r}) \;=\; e^{-\frac{V(\vec{r})-V(\vec{r}_0)}{k_BT}}n(\vec{r}_0)\,, \tag{17}$$

$$n(\vec{r}) \;=\; e^{-\frac{m\omega^2(\vec{r}^2-\vec{r}_0^2)}{2k_BT}}n(\vec{r}_0)\,, \tag{18}$$

the full solution is given by

$$f_{\rm eq}(\vec{r},\vec{p}) = n(\vec{0})\left(\frac{2\pi\hbar^2}{mk_BT}\right)^{3/2} e^{-\frac{1}{k_BT}\left(\frac{\vec{p}^2}{2m}+\frac{m\omega^2\vec{r}^2}{2}\right)} \tag{19}$$

ii) Show that for a generic function $g(\vec{r},\vec{p})$ of coordinates and momenta, the evolution of the average of this quantity

$$\langle g(\vec{r},\vec{p})\rangle \equiv \int \frac{d^3\vec{r}d^3\vec{p}}{(2\pi\hbar)^3} f(t,\vec{r},\vec{p})\; g(\vec{r},\vec{p})$$

is governed by

$$\frac{d\langle g(\vec{r},\vec{p})\rangle}{dt} - \left\langle \frac{\vec{p}}{m}\;\vec{\nabla}_{\vec{r}}g(\vec{r},\vec{p})\right\rangle - \left\langle \vec{F}(\vec{r})\;\vec{\nabla}_{\vec{p}}g(\vec{r},\vec{p})\right\rangle = \int \frac{d^3\vec{r}d^3\vec{p}}{(2\pi\hbar)^3}\; g(\vec{r},\vec{p})\; C[f](t,\vec{r},\vec{p}) \tag{20}$$

- The Boltzmann equation is written

$$\begin{aligned}
\int \frac{d^3\vec{r}d^3\vec{p}}{(2\pi\hbar)^3}\, g(\vec{r},\vec{p}) \left(\partial_t + \frac{\vec{p}}{m}\cdot\vec{\nabla}_r + \vec{F}\cdot\vec{\nabla}_p\right) f(t,\vec{r},\vec{p}) &= \int \frac{d^3\vec{r}d^3\vec{p}}{(2\pi\hbar)^3}\, g(\vec{r},\vec{p}) C[f](t,\vec{r},\vec{p})\,, \\
\frac{d\langle g(\vec{r},\vec{p})\rangle}{dt} - \int \frac{d^3\vec{r}d^3\vec{p}}{(2\pi\hbar)^3}\, f(t,\vec{r},\vec{p}) \left(\frac{\vec{p}}{m}\cdot\vec{\nabla}_r + \vec{F}\cdot\vec{\nabla}_p\right) g(\vec{r},\vec{p}) &= \int \frac{d^3\vec{r}d^3\vec{p}}{(2\pi\hbar)^3}\, g(\vec{r},\vec{p}) C[f](t,\vec{r},\vec{p})\,, \\
\frac{d\langle g(\vec{r},\vec{p})\rangle}{dt} - \left\langle \frac{\vec{p}}{m}\,\vec{\nabla}_{\vec{r}} g(\vec{r},\vec{p})\right\rangle - \left\langle \vec{F}(\vec{r})\,\vec{\nabla}_{\vec{p}} g(\vec{r},\vec{p})\right\rangle &= \int \frac{d^3\vec{r}d^3\vec{p}}{(2\pi\hbar)^3}\, g(\vec{r},\vec{p})\; C[f](t,\vec{r},\vec{p})\,,
\end{aligned} \tag{21}$$

iii) Explain why for $g(\vec{r},\vec{p}) = g_N(\vec{r}) + \vec{g}_{\vec{p}}(\vec{r})\vec{p} + g_e(\vec{r})\frac{\vec{p}^2}{2m}$ the right hand side of Eq. (20) vanishes irrespective of the spatial dependence of the coefficient functions $g_{N,\vec{p},e}(\vec{r})$.

- The right hand side is written (for $2 \leftrightarrow 2$ collision integral)

$$\mathcal{C}_{\rm coll}[f](t,\vec{r},\vec{p}_1) = \int \frac{d^3\vec{r}d^3\vec{p}_1}{(2\pi\hbar)^3}\, g(\vec{r},\vec{p}_1) \int_{\vec{p}_2}\int_{\vec{p}_3}\int_{\vec{p}_4} \tilde{w}(\vec{p}_1\vec{p}_2 \to \vec{p}_3\vec{p}_4)\,[f_3 f_4 - f_1 f_2]\,, \tag{22}$$

(23)

by renaming the integration variables one finds

$$\mathcal{C}_{\rm coll}[f](t,\vec{r},\vec{p}_1) = \frac{1}{4}\int_{\vec{r}}\int_{\vec{p}_1}\int_{\vec{p}_2}\int_{\vec{p}_3}\int_{\vec{p}_4} \tilde{w}(\vec{p}_1\vec{p}_2 \to \vec{p}_3\vec{p}_4)\,(g_1+g_2-g_3-g_4)\,[f_3 f_4 - f_1 f_2]\,, \tag{24}$$

where we used $f_i \equiv f(t,\vec{r},\vec{p}_i)$ and $g_i \equiv g(\vec{r},\vec{p}_i)$.

- Due to the conservation of particle number, momentum and energy, the collision integral for a function of the form $g(\vec{r},\vec{p}) = g_N(\vec{r}) + \vec{g}_{\vec{p}}(\vec{r})\vec{p} + g_e(\vec{r})\frac{\vec{p}^2}{2m}$ will vanish.

iv) Derive the explicit form for the equations of motion for the quantities $e_{\rm pot}(\vec{r},\vec{p}) = \frac{1}{2}m\omega^2\vec{r}^2$, $e_{\rm kin}(\vec{r},\vec{p}) = \frac{\vec{p}^2}{2m}$ and $e_{\rm corr}(\vec{r},\vec{p}) = \omega\frac{\vec{r}\cdot\vec{p}}{2}$.

- We have

$$\begin{aligned}
\frac{d\langle e_{\rm pot}(\vec{r},\vec{p})\rangle}{dt} - \left\langle \frac{\vec{p}}{m}\,\vec{\nabla}_{\vec{r}} e_{\rm pot}(\vec{r},\vec{p})\right\rangle - \left\langle \vec{F}(\vec{r})\,\vec{\nabla}_{\vec{p}} e_{\rm pot}(\vec{r},\vec{p})\right\rangle &= \int \frac{d^3\vec{r}d^3\vec{p}}{(2\pi\hbar)^3}\, e_{\rm pot}(\vec{r},\vec{p})\; C[f](t,\vec{r},\vec{p})\,, \\
\frac{d\langle e_{\rm pot}(\vec{r},\vec{p})\rangle}{dt} - \omega^2\left\langle \vec{p}\cdot\vec{r}\right\rangle &= 0\,,
\end{aligned} \tag{25}$$

$$\begin{aligned}
\frac{d\langle e_{\rm kin}(\vec{r},\vec{p})\rangle}{dt} - \left\langle \frac{\vec{p}}{m}\,\vec{\nabla}_{\vec{r}} e_{\rm kin}(\vec{r},\vec{p})\right\rangle - \left\langle \vec{F}(\vec{r})\,\vec{\nabla}_{\vec{p}} e_{\rm kin}(\vec{r},\vec{p})\right\rangle &= \int \frac{d^3\vec{r}d^3\vec{p}}{(2\pi\hbar)^3}\, e_{\rm pot}(\vec{r},\vec{p})\; C[f](t,\vec{r},\vec{p})\,, \\
\frac{d\langle e_{\rm kin}(\vec{r},\vec{p})\rangle}{dt} + \omega^2\left\langle \vec{r}\cdot\vec{p}\right\rangle &= 0\,,
\end{aligned} \tag{26}$$

$$\begin{aligned}
\frac{d\langle e_{\rm corr}(\vec{r},\vec{p})\rangle}{dt} - \left\langle \frac{\vec{p}}{m}\,\vec{\nabla}_{\vec{r}}\, e_{\rm corr}(\vec{r},\vec{p})\right\rangle - \left\langle \vec{F}(\vec{r})\,\vec{\nabla}_{\vec{p}}\, e_{\rm corr}(\vec{r},\vec{p})\right\rangle &= \int \frac{d^3\vec{r}d^3\vec{p}}{(2\pi\hbar)^3}\, e_{\rm pot}(\vec{r},\vec{p})\; C[f](t,\vec{r},\vec{p})\,, \\
\frac{d\langle e_{\rm corr}(\vec{r},\vec{p})\rangle}{dt} - \frac{\omega}{2}\left\langle \frac{\vec{p}^2}{m}\right\rangle + \frac{m\omega^3}{2}\left\langle \vec{r}^2\right\rangle &= 0\,,
\end{aligned} \tag{27}$$

v) Based on your results in (iv) show that the Boltzmann gas in a harmonic trap can exhibit oscillatory behavior in the long time limit, and therefore does not relax towards the global equilibrium solution. Determine the frequency of oscillations.

- The evolution equation are given by

$$\frac{d\langle e_{\rm pot}(\vec{r},\vec{p})\rangle}{dt} = 2\omega\langle e_{\rm corr}(\vec{r},\vec{p})\rangle \,, \tag{28}$$

$$\frac{d\langle e_{\rm kin}(\vec{r},\vec{p})\rangle}{dt} = -2\omega\langle e_{\rm corr}(\vec{r},\vec{p})\rangle \,, \tag{29}$$

$$\frac{d\langle e_{\rm corr}(\vec{r},\vec{p})\rangle}{dt} = \omega\langle e_{\rm kin}(\vec{r},\vec{p})\rangle - \omega\langle e_{\rm pot}(\vec{r},\vec{p})\rangle \,, \tag{30}$$

$$\frac{d^2\langle e_{\rm pot}(\vec{r},\vec{p})\rangle}{dt^2} = 2\omega^2\langle e_{\rm kin}(\vec{r},\vec{p})\rangle - 2\omega^2\langle e_{\rm pot}(\vec{r},\vec{p})\rangle = -4\omega^2\langle e_{\rm pot}(\vec{r},\vec{p}) + 2\omega^2\langle e_{\rm tot}(\vec{r},\vec{p})\rangle \,, \tag{31}$$

$$\frac{d\langle e_{\rm kin}(\vec{r},\vec{p})\rangle}{dt} = -2\omega^2\langle e_{\rm kin}(\vec{r},\vec{p})\rangle + 2\omega^2\langle e_{\rm pot}(\vec{r},\vec{p})\rangle = -4\omega^2\langle e_{\rm kin}(\vec{r},\vec{p}) + 2\omega^2\langle e_{\rm tot}(\vec{r},\vec{p})\rangle \,, \tag{32}$$

$$e_{\rm pot}(\vec{r},\vec{p}) = e_{\rm tot}(\vec{r},\vec{p}) - e_{\rm kin}(\vec{r},\vec{p}) = c_0 + c_1\cos(2\omega t) + c_2\sin(2\omega t) \,, \tag{33}$$

The frequency of oscillations is 2ω.

3.2 Electric conductivity & eff. relaxation time of a Lorentz gas

Consider a dilute gas of light particles of mass m and heavy particles of mass M, dominated by elastic interactions between light and heavy particles. Since the kinetic motion of heavy particles is suppressed by their large mass, they can be described as static scattering centers; the dynamics of the light particles is then governed by the kinetic equation for a Lorentz gas

$$\left(\frac{\partial}{\partial t} + \frac{\vec{p}}{m}\vec{\nabla}_{\vec{r}} + \vec{F}\vec{\nabla}_{\vec{p}}\right) f_{\rm light}(t,\vec{r},\vec{p}) = C[f_{\rm light}](t,\vec{r},\vec{p}) \tag{34}$$

$$C[f_{\rm light}](t,\vec{r},\vec{p}) = n_{\rm heavy}\frac{|\vec{p}|}{m}\int d\Omega_{\vec{p}\vec{p}'}\frac{d\sigma}{d\Omega_{\vec{p}\vec{p}'}}(\vec{p}\to\vec{p}')\left[f_{\rm light}(t,\vec{r},\vec{p}') - f_{\rm light}(t,\vec{r},\vec{p})\right] \tag{35}$$

where $n_{\rm heavy}$ denotes the (uniform) density of heavy particles in the system, $\Omega_{\vec{p},\vec{p}'}$ is the scattering angle and $\frac{d\sigma}{d\Omega}(\vec{p}\to\vec{p}')$ denotes the cross-section for the interaction.

We will assume that the interactions are elastic and particle number conserving, i.e. the number of light particles is conserved and the energy $\Delta E = \frac{(\vec{p}-\vec{p}')^2}{2M}$ transferred to the heavy particles is negligible. Nevertheless, momentum can be transferred from light to heavy particles, i.e. the differential cross-section $\frac{d\sigma}{d\Omega}(\vec{p}\to\vec{p}')$ is non-zero even when $\vec{p}\neq\vec{p}'$.

i) Show that local equilibrium solutions for $f = f_{\rm light}$ are of the form

$$f^{(0)}(t,\vec{r},\vec{p}) = \exp\left(-\frac{\epsilon_{\vec{p}} - \mu(t,\vec{r})}{k_B T(t,\vec{r})}\right) \,, \qquad \epsilon_{\vec{p}} = \vec{p}^2/2m \,. \tag{36}$$

What differences do you observe in comparison to local equilibrium solutions of the Boltzmann equation for two-body interactions between light particles?

- Local equilibrium solution cancels the collision integral on the RHS of the Boltzmann equation, i.e

$$f^{(0)}(t,\vec{r},\vec{p}') - f^{(0)}(t,\vec{r},\vec{p}) = 0 \,, \tag{37}$$

taking the logarithm following the lecture

$$\log f^{(0)}(t,\vec{r},\vec{p}') = \log f^{(0)}(t,\vec{r},\vec{p})\ . \tag{38}$$

We find that $\log f^{(0)}(t,\vec{r},\vec{p})$ is a collision invariant, only particle number and energy are considered invariants.

$$\log f^{(0)}(t,\vec{r},\vec{p}) = \lambda_N + \lambda_\epsilon \frac{\vec{p}^2}{2m}\ , \tag{39}$$

$$f^{(0)}(t,\vec{r},\vec{p}) = \exp\left(-\frac{\frac{\vec{p}^2}{2m} - \mu(t,\vec{r})}{k_B T(t,\vec{r})}\right)\ , \tag{40}$$

where we match the constant to obtain the ideal gas equation of state. Compared to the local equilibrium solution of Boltzmann equation for two-body interactions, the velocity term is missing because here momentum is not conserved.

We will assume in the following that the differential cross section $\frac{d\sigma}{d\Omega}(\vec{p}\to\vec{p}')$ is a function of of the magnitude of the momentum $|\vec{p}| = |\vec{p}'|$ and the scattering angle $\theta_{pp'}$ only. We now consider the effect of a constant external electric field $\vec{E}$ in the limit where the change in velocity due to the Lorentz force between individual collisions is small compared to the thermal velocity $\frac{q|\vec{E}|}{m}\tau_{\rm mfp} \ll v_{\rm th}$.

ii) Demonstrate that to leading order in $\frac{q|\vec{E}|}{mv_{\rm th}}\tau_{\rm mfp} \ll 1$, the stationary solutions to the Boltzmann equation for a spatially homoegenous Lorentz gas are given by $f(\vec{p}) = f^{(0)}(\vec{p}) + f^{(1)}(\vec{p})$, where $f^{(0)}(\vec{p})$ is the local equilibrium distribution and $f^{(1)}(\vec{p})$ is determined by

$$\delta C[f^{(1)}](\vec{p}) = -q\frac{\vec{p}\cdot\vec{E}}{mk_BT} f^{(0)}(\vec{p})\ , \tag{41}$$

where

$$\delta C[f^{(1)}](\vec{p}) = n_{\rm heavy}\frac{|\vec{p}|}{m}\int d\Omega_{\vec{p}\vec{p}'}\frac{d\sigma}{d\Omega_{\vec{p}\vec{p}'}}(\vec{p}\to\vec{p}')\left[f^{(1)}(\vec{p}') - f^{(1)}(\vec{p})\right] \tag{42}$$

- For a stationary and spatially homoegenous Lorentz gas. i.e. $T(t,\vec{r}) = T$ and $\mu(t,\vec{r}) = \mu$, such that at leading order in $\frac{q|\vec{E}|}{mv_{\rm th}}\tau_{\rm mfp} \ll 1$, we have

$$\left(\frac{\partial}{\partial t} + \frac{\vec{p}}{m}\vec{\nabla}_{\vec{r}} + \vec{F}\vec{\nabla}_{\vec{p}}\right)\left(f^{(0)}(\vec{p}) + f^{(1)}(\vec{p})\right) = \delta C[f^{(1)}](t,\vec{r},\vec{p})\ , \tag{43}$$

$$-q\frac{\vec{p}\cdot\vec{E}}{mk_BT} f^{(0)}(\vec{p}) = \delta C[f^{(1)}](t,\vec{r},\vec{p})\ . \tag{44}$$

Based on the relaxation time approximation (RTA), the linearized collision operator $\delta C[f^{(1)}]$ is approximated as

$$\delta C[f^{(1)}](\vec{p})\Big|_{RTA} = -\frac{f^{(1)}(\vec{p})}{\tau_R(\epsilon_{\vec{p}})} \tag{45}$$

with an energy dependent relaxation time $\tau_R(\epsilon_{\vec{p}})$ and the functional form of the distribution $f^{(1)}$ is given by

$$f^{(1)}(\vec{p})\Big|_{RTA} = q\tau_R(\epsilon_{\vec{p}})\frac{\vec{p}\cdot\vec{E}}{mk_BT} f^{(0)}(\vec{p})\ . \tag{46}$$

iii) Show that by equating δC in (42) and (45) and using the solution in (46) for the functional form of the distribution $f^{(1)}$ the energy dependent relaxation time $\tau_R(\epsilon_{\vec{p}})$ can be determined self-consistently according to

$$\frac{1}{\tau_R(\epsilon_{\vec{p}})} = 2\pi n_{\text{heavy}} \frac{|\vec{p}|}{m} \int d\cos(\theta_{pp'}) \frac{d\sigma}{d\Omega_{pp'}} \Big[1 - \cos(\theta_{pp'})\Big] \tag{47}$$

$\Big($Hint: By appropriate choice of coordinates you can express $\vec{p} \cdot \vec{E} = |\vec{p}||\vec{E}|\cos(\theta_p)$ and $\vec{p}' \cdot \vec{E} = |\vec{p}'||\vec{E}|\cos(\theta_p)\cos(\theta_{pp'}) + \sin(\theta_p)\sin(\theta_{pp'})\cos(\phi_{p'})$ $\Big)$

- The collision integral is given by

$$\begin{aligned}
\delta C[f^{(1)}](\vec{p}) &= n_{\text{heavy}} \frac{|\vec{p}|}{m} \int d\Omega_{\vec{p}\vec{p}'} \frac{d\sigma}{d\Omega_{\vec{p}\vec{p}'}}(\vec{p} \to \vec{p}') \Big[f^{(1)}(\vec{p}') - f^{(1)}(\vec{p})\Big] && (48)\\
-\frac{f^{(1)}(\vec{p})}{\tau_R(\epsilon_{\vec{p}})} &= n_{\text{heavy}} \frac{|\vec{p}|}{m} \int d\Omega_{\vec{p}\vec{p}'} \frac{d\sigma}{d\Omega_{\vec{p}\vec{p}'}} \Big[f^{(1)}(\vec{p}') - f^{(1)}(\vec{p})\Big] && (49)\\
-\frac{\vec{p} \cdot \vec{E}}{\tau_R(\epsilon_{\vec{p}})} &= n_{\text{heavy}} \frac{|\vec{p}|}{m} \int d\cos\theta_{pp'} d\phi_{pp'} \frac{d\sigma}{d\Omega_{\vec{p}\vec{p}'}} \left[\vec{E} \cdot \vec{p}' \exp\left(-\frac{\epsilon_{\vec{p}'} - \epsilon_{\vec{p}}}{k_B T(t,\vec{r})}\right) - \vec{E} \cdot \vec{p}\right] && (50)\\
\frac{1}{\tau_R(\epsilon_{\vec{p}})} &= 2\pi n_{\text{heavy}} \frac{|\vec{p}|}{m} \int d\cos\theta_{pp'} \frac{d\sigma}{d\Omega_{\vec{p}\vec{p}'}} \left[1 - \cos(\theta_{pp'})\right] && (51)
\end{aligned}$$

iv) Determine the energy dependent relaxation time $\tau_R(\epsilon_{\vec{p}})$ for the scattering off hard-sphere scattering centers $\frac{d\sigma}{d\Omega_{pp'}} = \frac{R^2}{4}$ and calculate the electrical conductivity σ_{el} for this model.

- The relaxation time is given by

$$\begin{aligned}
\frac{1}{\tau_R(\epsilon_{\vec{p}})} &= 2\pi n_{\text{heavy}} \frac{|\vec{p}|}{m} \int d\cos\theta_{pp'} \frac{R^2}{4} \left[1 - \cos(\theta_{pp'})\right] , && (52)\\
&= \pi n_{\text{heavy}} \frac{|\vec{p}|}{m} && (53)
\end{aligned}$$

- The electrical current is given by

$$\begin{aligned}
\vec{J}_{el} &= q \int_{\vec{p}} \frac{p^i}{m} f^{(1)}(\vec{p}) , && (54)\\
&= \frac{q^2}{m^2 k_B T} \int_{\vec{p}} \vec{p}\, \tau_R(\epsilon_{\vec{p}})(\vec{p} \cdot \vec{E}) f^{(0)}(\vec{p}) , && (55)\\
\vec{J}_{el} &= \sigma_{el}^{ij} \vec{E}^j . && (56)
\end{aligned}$$

Giving the expression for the electrical conductivity

$$\begin{aligned}
\sigma^{ij} &= \frac{q^2}{m k_B T \pi n_{\text{heavy}}} \int_{\vec{p}} \frac{1}{|\vec{p}|} p^i p^j f^{(0)}(\vec{p}) , && (57)\\
&= q^2 \frac{16 (m k_B T)^3}{(\hbar\pi)^3 n_{\text{heavy}}} e^{\left(\frac{\mu}{k_B T}\right)} \delta^{ij} . && (58)
\end{aligned}$$