Non-equilibrium physics WS 20/21 – Exercise Sheet 7:

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1 Discussion:

i) What are the differences between the collisions integral of the Boltzmann equation for classical and quantum systems?

- The collisions integral of classical Boltzmann equation is written

$$\mathcal{C}_{\text{coll}}[f](t,\vec{r},\vec{p}_1) = \int_{\vec{p}_2}\int_{\vec{p}_3}\int_{\vec{p}_4} \tilde{w}(\vec{p}_1\vec{p}_2 \to \vec{p}_3\vec{p}_4)\,[f(\vec{p}_3)f(\vec{p}_4) - f(\vec{p}_1)f(\vec{p}_2)] \;, \tag{1}$$

it describes how the distribution gains particles with momentum $\vec{p}_1$ due to scattering between $\vec{p}_3, \vec{p}_4$; and the loss of particles with momentum $\vec{p}_1$ when scattering with particle with momenutm $\vec{p}_2$.

- The collisions integral of quantum Boltzmann equation is written

$$\begin{aligned}\mathcal{C}_{\text{coll}}[f](t,\vec{r},\vec{p}_1) = &\int_{\vec{p}_2}\int_{\vec{p}_3}\int_{\vec{p}_4} \tilde{w}(\vec{p}_1\vec{p}_2 \to \vec{p}_3\vec{p}_4)\,[f(\vec{p}_3)f(\vec{p}_4)(1\pm f(\vec{p}_1))(1\pm f(\vec{p}_2)) \\ &-f(\vec{p}_1)f(\vec{p}_2)(1\pm f(\vec{p}_3))(1\pm f(\vec{p}_4))] \;, \end{aligned}\tag{2}$$

In contrast to the classical case, we have to consider if a particle with momentum $\vec{p}_i$ is present in the system which due to Bose enhancement/Fermi blocking, will either enhance or block the process. This is taken into account with the terms $(1 \pm f(\vec{p}_i))$.

ii) What is the physical meaning of balance equations, and how can the fluxes of particle number and momentum be understood from a microscopic perspective?

- Balance equations determine the local conservation laws

$$\underbrace{\frac{\partial g(t,\vec{r})}{\partial t}}_{\text{Local change}} + \underbrace{\vec{\nabla}\cdot\vec{J}_G(t,\vec{r})}_{\text{Flux}} = \underbrace{\sigma_G(t,\vec{r})}_{\text{Creation}} \tag{3}$$

- The flux of particle number is given by

$$\vec{J}_N(t,\vec{r}) = \int \frac{d^3\vec{p}}{(2\pi\hbar)^3} \underbrace{\frac{\vec{p}}{m}}_{\text{velocity}} \underbrace{f(t,\vec{r},\vec{p})}_{\text{probability}} \;, \tag{4}$$

- The flux of momentum is given by

$$\vec{J}_{\vec{P}}(t,\vec{r}) = \int \frac{d^3\vec{p}}{(2\pi\hbar)^3} \underbrace{\frac{\vec{p}}{m}}_{\text{velocity}} \otimes \underbrace{\vec{p}}_{\text{momentum}} f(t,\vec{r},\vec{p}) \;, \tag{5}$$

2 In-class problems:

2.1 Energy balance in the Boltzmann equation

i) Derive the balance equation for the kinetic energy density

$$e_{\text{kin}}(t,\vec{r}) = \int \frac{d^3\vec{p}}{(2\pi\hbar)^3} \frac{\vec{p}^2}{2m} f(t,\vec{r},\vec{p}) ,$$

and show that the energy flux J_E is given by

$$J_{E_{\text{kin}}}(t,\vec{r}) = \int \frac{d^3\vec{p}}{(2\pi\hbar)^3} \frac{\vec{p}^2}{2m} \frac{\vec{p}}{m} f(t,\vec{r},\vec{p}) ,$$

- The Boltzmann equation is written

$$\left(\partial_t + \frac{\vec{p}}{m}\cdot\vec{\nabla}_r\right) f(t,\vec{r},\vec{p}) = C[f](t,\vec{r},\vec{p}) . \tag{6}$$

Taking the energy moment leads to

$$\int \frac{d^3\vec{p}}{(2\pi\hbar)^3} \frac{\vec{p}^2}{2m} \left(\partial_t + \frac{\vec{p}}{m}\cdot\vec{\nabla}_r + \vec{F}\cdot\vec{\nabla}_p\right) f(t,\vec{r},\vec{p}) = \int \frac{d^3\vec{p}}{(2\pi\hbar)^3} \frac{\vec{p}^2}{2m} C[f](t,\vec{r},\vec{p}) ,$$

$$\partial_t\epsilon_{\text{kin}} + \vec{\nabla}_r\cdot\int \frac{d^3\vec{p}}{(2\pi\hbar)^3} \frac{\vec{p}^2}{2m} \frac{\vec{p}}{m} f(t,\vec{r},\vec{p}) = -\vec{F}\cdot\int \frac{d^3\vec{p}}{(2\pi\hbar)^3} \left(\frac{\vec{p}^2}{2m}\right) \vec{\nabla} p f(t,\vec{r},\vec{p}) , \tag{7}$$

$$\partial_t\epsilon_{\text{kin}} + \vec{\nabla}_r\cdot\int \frac{d^3\vec{p}}{(2\pi\hbar)^3} \frac{\vec{p}^2}{2m} \frac{\vec{p}}{m} f(t,\vec{r},\vec{p}) = \vec{F}\cdot\int \frac{d^3\vec{p}}{(2\pi\hbar)^3} \vec{\nabla} p \left(\frac{\vec{p}^2}{2m}\right) f(t,\vec{r},\vec{p}) , \tag{8}$$

$$\partial_t\epsilon_{\text{kin}} + \vec{\nabla}_r\cdot\int \frac{d^3\vec{p}}{(2\pi\hbar)^3} \frac{\vec{p}^2}{2m} \frac{\vec{p}}{m} f(t,\vec{r},\vec{p}) = \vec{F}\cdot\int \frac{d^3\vec{p}}{(2\pi\hbar)^3} \frac{\vec{p}}{m} f(t,\vec{r},\vec{p}) , \tag{9}$$

$$\tag{10}$$

where the collision integral part without external forces conserves energy leading to a vanishing contribution. We can rewrite this by introducing the energy flux

$$J_{E_{\text{kin}}}(t,\vec{r}) = \int \frac{d^3\vec{p}}{(2\pi\hbar)^3} \frac{\vec{p}^2}{2m} \frac{\vec{p}}{m} f(t,\vec{r},\vec{p}) , \tag{11}$$

$$\partial_t\epsilon_{\text{kin}} + \vec{\nabla}_r J_{E_{\text{kin}}}(t,\vec{r}) = \vec{F}\cdot\vec{J}_N(t,\vec{r}) , \tag{12}$$

ii) Calculate the particle number flux $J_N(t,\vec{r})$, the kinetic energy flux $J_{E_{\text{kin}}}(t,\vec{r})$ and the momentum flux $J_{\vec{P}}(t,\vec{r})$ for an ideal gas in local thermal equilibrium, where

$$f(t,\vec{r},\vec{p}) = f^{(0)}(t,\vec{r},\vec{p}) = n(t,\vec{r}) \left(\frac{2\pi\hbar^2}{mk_BT(t,\vec{r})}\right)^{3/2} \exp\left[-\frac{\left(\vec{p}-m\vec{v}(t,\vec{r})\right)^2}{2mk_BT(t,\vec{r})}\right] .$$

- The particle number flux $J_N(t,\vec{r})$ is written

$$\vec{J}_N(t,\vec{r}) = \int \frac{d^3\vec{p}}{(2\pi\hbar)^3} \frac{\vec{p}}{m} f^{(0)}(t,\vec{r},\vec{p}) , \tag{13}$$

$$= \int \frac{d^3\vec{p}}{(2\pi\hbar)^3} \frac{\vec{p}-mv(t,\vec{r})}{m} f^{(0)}(t,\vec{r},\vec{p}) + v(t,\vec{r}) \int \frac{d^3\vec{p}}{(2\pi\hbar)^3} f^{(0)}(t,\vec{r},\vec{p}) \tag{14}$$

$$= n(t,\vec{r})\vec{v}(t,\vec{r}) \tag{15}$$

- The kinetic energy flux $J_{E_{\rm kin}}(t,\vec{r})$ is written

$$\begin{aligned}
\vec{J}_{E_{\rm kin}}(t,\vec{r}) &= \int \frac{d^3\vec{p}}{(2\pi\hbar)^3}\,\frac{\vec{p}^2}{2m}\frac{\vec{p}}{m}\,f^{(0)}(t,\vec{r},\vec{p})\;, && (16)\\
&= \frac{1}{2}\Big[3n(t,\vec{r})k_BT(t,\vec{r}) + mn(t,\vec{r})\vec{v}^2(t,\vec{r})\Big]\vec{v}(t,\vec{r}) + n(t,\vec{r})k_BT(t,\vec{r})\vec{v}(t,\vec{r})\;, && (17)\\
&= \left[\epsilon + \frac{1}{2}\rho(t,\vec{r})\vec{v}^2(t,\vec{r})\right]\vec{v}(t,\vec{r}) + P(t,\vec{r})\vec{v}(t,\vec{r})\;, && (18)\\
& && (19)
\end{aligned}$$

- The momentum flux $J_{\vec{P}}(t,\vec{r})$ is written

$$\begin{aligned}
J^{ij}_{\vec{P}}(t,\vec{r}) &= \int \frac{d^3\vec{p}}{(2\pi\hbar)^3}\,\frac{\vec{p}\otimes\vec{p}}{m}\,f^{(0)}(t,\vec{r},\vec{p})\;, && (20)\\
J^{ij}_{\vec{P}}(t,\vec{r}) &= \int \frac{d^3\vec{p}}{(2\pi\hbar)^3}\,\frac{(\vec{p}+m\vec{v})\otimes(\vec{p}+m\vec{v})}{m}\,f^{(0)}(t,\vec{r},\vec{p}+m\vec{v})\;, && (21)\\
&= mn(t,\vec{r})\vec{v}(t,\vec{r})\otimes\vec{v}(t,\vec{r}) + \delta^{ij}n(t,\vec{r})k_BT(t,\vec{r}) && (22)\\
&= \rho(t,\vec{r})\vec{v}(t,\vec{r})\otimes\vec{v}(t,\vec{r}) + \delta^{ij}P(t,\vec{r}) && (23)
\end{aligned}$$

Since we are considering an equilibrium system, we recognize that these are the constitutive equations of an ideal fluid.

3 Homework problems:

3.1 Linearized Boltzmann equation

Consider the Boltzmann equation for neutral particles with mass m in absence of external forces.

i) Show that the Maxwell–Boltzmann distribution

$$f^{(0)}(\vec{p}) = n \left(\frac{2\pi\hbar^2}{mk_BT}\right)^{3/2} \exp\left[-\frac{\vec{p}^2}{2mk_BT}\right]$$

with constant temperature T and density n is a stationary solution of the Boltzmann equation.

- The Boltzmann equation is written

$$\left(\frac{\partial}{\partial t} + \frac{\vec{p}_1}{m}\vec{\nabla}_{\vec{r}}\right) f(t,\vec{r},\vec{p}_1) = \mathcal{C}_{\text{coll}}[f](t,\vec{r},\vec{p}_1) \;, \tag{24}$$

$$\tag{25}$$

the left-hand side vanishes as the distribution is constant and does not depend on space. The collision integral is written

$$\begin{aligned}
\mathcal{C}_{\text{coll}}[f](t,\vec{r},\vec{p}_1) &= \int_{\vec{p}_2}\int_{\vec{p}_3}\int_{\vec{p}_4} \tilde{w}(\vec{p}_1\vec{p}_2 \to \vec{p}_3\vec{p}_4)\left[f^{(0)}(\vec{p}_3)f^{(0)}(\vec{p}_4) - f^{(0)}(\vec{p}_1)f^{(0)}(\vec{p}_2)\right] \;, & (26)\\
&\propto \int_{\vec{p}_2}\int_{\vec{p}_3}\int_{\vec{p}_4} \tilde{w}(\vec{p}_1\vec{p}_2 \to \vec{p}_3\vec{p}_4)\left[\exp\left[-\frac{\vec{p}_3^2+\vec{p}_4^2}{2mk_BT}\right] - \exp\left[-\frac{\vec{p}_1^2+\vec{p}_2^2}{2mk_BT}\right]\right] \;, & (27)\\
&= 0 \;, & (28)
\end{aligned}$$

which vanishes due to energy conservation $\vec{p}_1^2+\vec{p}_2^2 = \vec{p}_3^2+\vec{p}_4^2$ (particles have the same mass).

Next we will consider small perturbations away from the equilibrium solution, which will be characterized by a function $h(t,\vec{r},\vec{p})$ according to

$$f(t,\vec{r},\vec{p}) = f^{(0)}(\vec{p})\left[1 + h(t,\vec{r},\vec{p})\right] \;.$$

and we will neglect terms of $\mathcal{O}(h^2)$ and higher in the following.

ii) Show that that the linearized evolution equation for h is given by

$$\left(\frac{\partial}{\partial t} + \frac{\vec{p}_1}{m}\vec{\nabla}_{\vec{r}}\right) h(t,\vec{r},\vec{p}_1) = \mathcal{I}_{\text{coll}}[h](t,\vec{r},\vec{p}_1)$$

where $\mathcal{I}_{\text{coll}}[h](t,\vec{r},\vec{p}_1)$ denotes (linear) collisions operator

$$\mathcal{I}_{\text{coll}}[h](t,\vec{r},\vec{p}_1) = \int_{\vec{p}_2}\int_{\vec{p}_3}\int_{\vec{p}_4} \tilde{w}(\vec{p}_1\vec{p}_2 \to \vec{p}_3\vec{p}_4) f^{(0)}(\vec{p}_2)\left[h(t,\vec{r},\vec{p}_3) + h(t,\vec{r},\vec{p}_4) - h(t,\vec{r},\vec{p}_1) - h(t,\vec{r},\vec{p}_2)\right]$$

(Hint: Detailed balance in equilibrium guarantees $f^{(0)}(\vec{p}_1)f^{(0)}(\vec{p}_2) = f^{(0)}(\vec{p}_3)f^{(0)}(\vec{p}_4)$ for all combinations of momenta allowed by energy conservation.)

- The Boltzmann equation is given by

$$\begin{aligned}
\left(\frac{\partial}{\partial t} + \frac{\vec{p}_1}{m}\vec{\nabla}_{\vec{r}}\right) f^{(0)}(\vec{p})\left[1 + h(t,\vec{r},\vec{p})\right] &= \mathcal{I}_{\text{coll}}[h](t,\vec{r},\vec{p}_1) \;, & (29)\\
f^{(0)}(\vec{p})\left(\frac{\partial}{\partial t} + \frac{\vec{p}_1}{m}\vec{\nabla}_{\vec{r}}\right) h(t,\vec{r},\vec{p}) &= f^{(0)}(\vec{p})\mathcal{I}_{\text{coll}}[h](t,\vec{r},\vec{p}_1) \;, & (30)\\
& & (31)
\end{aligned}$$

$$\mathcal{I}_{\text{coll}}[h](t,\vec{r},\vec{p}_1) = \frac{1}{f^{(0)}(\vec{p})}\int_{\vec{p}_2}\int_{\vec{p}_3}\int_{\vec{p}_4}\tilde{w}(\vec{p}_1\vec{p}_2\to\vec{p}_3\vec{p}_4)\Big[f^{(0)}(\vec{p}_3)f^{(0)}(\vec{p}_4)\left[1+h(t,\vec{r}_3,\vec{p}_3)\right]\left[1+h(t,\vec{r}_4,\vec{p}_4)\right] - f^{(0)}(\vec{p}_1)f^{(0)}(\vec{p}_2)\left[1+h(t,\vec{r}_2,\vec{p}_2)\right]\left[1+h(t,\vec{r}_1,\vec{p}_1)\right]\Big] \,, \tag{32}$$

$$= \frac{1}{f^{(0)}(\vec{p})}\int_{\vec{p}_2}\int_{\vec{p}_3}\int_{\vec{p}_4}\tilde{w}(\vec{p}_1\vec{p}_2\to\vec{p}_3\vec{p}_4)\Big[f^{(0)}(\vec{p}_3)f^{(0)}(\vec{p}_4)\left[h(t,\vec{r}_3,\vec{p}_3)+h(t,\vec{r}_4,\vec{p}_4)\right] - f^{(0)}(\vec{p}_1)f^{(0)}(\vec{p}_2)\left[h(t,\vec{r}_2,\vec{p}_2)+h(t,\vec{r}_1,\vec{p}_1)\right]\Big] \,, \tag{33}$$

$$= \int_{\vec{p}_2}\int_{\vec{p}_3}\int_{\vec{p}_4}\tilde{w}(\vec{p}_1\vec{p}_2\to\vec{p}_3\vec{p}_4)f^{(0)}(\vec{p}_2)\left[h(t,\vec{r}_3,\vec{p}_3)+h(t,\vec{r}_4,\vec{p}_4)-h(t,\vec{r}_2,\vec{p}_2)-h(t,\vec{r}_1,\vec{p}_1)\right] \,,$$

We now specialize on homogenous perturbations, where $h(t,\vec{r},\vec{p}_1) = h(t,\vec{p}_1)$ is independent of the spatial coordinate $\vec{r}$. In order to investigate their behavior, we follow the usual strategy and search for eigenfunctions $h_i(\vec{p}_1)$ of the collisions operator $\mathcal{I}_{\text{coll}}$, defined by the condition

$$\mathcal{I}_{\text{coll}}[h_i](\vec{p}_1) = \lambda_i h_i(\vec{p}_1) \tag{34}$$

where i labels the different eigenfunctions. We will assume that all eigenfunctions can be normalized according to

$$\int_{\vec{p}_1} f^{(0)}(\vec{p}_1)\Big[h_i(\vec{p}_1)\Big]^2 = 1$$

iii) Show that $\lambda = 0$ is a five fold degenerate eigenvalue and write down the corresponding unnormalized eigenfunctions $h_1(\vec{p}_1),\cdots,h_5(\vec{p}_1)$.
$\Big($Hint: Detailed properties of the transition rates $\tilde{w}(\vec{p}_1\vec{p}_2\to\vec{p}_3\vec{p}_4)$ are irrelevant to answer this questions – instead the eigenfunctions are determined by basic fundamental properties of two-body collisions. $\Big)$

- The collision integral conserves particle number, energy and momentum leading to $\lambda = 0$ being a five fold degenerate eigenvalue.
- if $\chi(1) \equiv \chi(t,\vec{r}_1,\vec{p}_1)$ is a collisional invariant one finds

$$\frac{\partial}{\partial t}\int_{\vec{p}_1} f^{(0)}(\vec{p})h_i(\vec{p}_1) = \int_{\vec{p}_1} f^{(0)}(\vec{p})\lambda_1 h_i(\vec{p}_1) = 0 \tag{35}$$

$$= \int_{\vec{p}_1} f^{(0)}(\vec{p})\mathcal{I}_{\text{coll}}[h](t,\vec{r},\vec{p}_1) \tag{36}$$

$$= \int_{\vec{p}_1}\int_{\vec{p}_2}\int_{\vec{p}_3}\int_{\vec{p}_4}\tilde{w}(\vec{p}_1\vec{p}_2\to\vec{p}_3\vec{p}_4)f^{(0)}(\vec{p}_1)f^{(0)}(\vec{p}_2)\left[h(\vec{p}_3)+h(\vec{p}_4)-h(\vec{p}_2)-h(\vec{p}_1)\right] \,, \tag{37}$$

$$= \int_{\vec{p}_1}\int_{\vec{p}_2}\int_{\vec{p}_3}\int_{\vec{p}_4}\tilde{w}(\vec{p}_1\vec{p}_2\to\vec{p}_3\vec{p}_4)f^{(0)}(\vec{p}_1)f^{(0)}(\vec{p}_2)\left[\chi(1)+\chi(2)-\chi(3)-\chi(4)\right] \,, \tag{38}$$

- For number conservation we have $\chi(1) = 1$

$$h_n(\vec{p}) = c = \frac{1}{\sqrt{\int_{\vec{p}_1} f^{(0)}(\vec{p}_1)}} = \frac{1}{\sqrt{n^{(0)}}} \tag{39}$$

- For energy conservation we write $\chi(1) = \frac{\vec{p}_1^2}{2m} - \frac{3}{2}k_BT$

$$h_\epsilon(\vec{p}) = c\left(\frac{\vec{p}_1^2}{2m}-\frac{3}{2}k_BT\right) = \left(\frac{\vec{p}_1^2}{2m}-\frac{3}{2}k_BT\right)\frac{1}{\sqrt{\int_{\vec{p}_1}\left(\frac{\vec{p}^2}{2m}-\frac{3}{2}k_BT\right)^2 f^{(0)}(\vec{p}_1)}} = \sqrt{\frac{2}{3n}}\left(\frac{\vec{p}^2}{2mk_BT}-\frac{3}{2}\right) \tag{40}$$

- For momentum conservation we have $\chi(1) = \vec{p}_1$

$$h_{\vec{p}}(\vec{p}) = c\vec{p} = \frac{\vec{p}}{\sqrt{\int_{\vec{p}_1} \vec{p}^2 f^{(0)}(\vec{p}_1)}} = \frac{\vec{p}}{\sqrt{mnk_BT}} \tag{41}$$

iv) Show that all other eigenvalues λ_i are negative $\lambda_i \leq 0$.
$\Big($ Hint: Exploit the normalization condition to express the eigenvalue λ_i as an integral moment of the collisions operator $\int_{\vec{p}_1} f^{(0)}(\vec{p}_1)h_i(\vec{p}_1)\mathcal{I}_{\text{coll}}[h_i](\vec{p}_1)$. Exploit the symmetries of the integrand to show that it is negative definite.$\Big)$

- Using the normalization condition we write

$$\begin{aligned}
\lambda_i &= \lambda_i \int_{\vec{p}_1} f^{(0)}(\vec{p}_1)h_i^2(\vec{p}_1) = \int_{\vec{p}_1} f^{(0)}(\vec{p}_1)h_i(\vec{p}_1)\mathcal{I}_{\text{coll}}[h_i](\vec{p}_1)\ , \qquad (42)\\
&= \int_{\vec{p}_1}\int_{\vec{p}_2}\int_{\vec{p}_3}\int_{\vec{p}_4} \tilde{w}(\vec{p}_1\vec{p}_2 \to \vec{p}_3\vec{p}_4) f^{(0)}(\vec{p}_1) f^{(0)}(\vec{p}_2) h_i(\vec{p}_1)\left[h_i(\vec{p}_3) + h_i(\vec{p}_4) - h_i(\vec{p}_2) - h_i(\vec{p}_1)\right]\\
&= \frac{1}{2}\int_{\vec{p}_1}\int_{\vec{p}_2}\int_{\vec{p}_3}\int_{\vec{p}_4} \tilde{w}(\vec{p}_1\vec{p}_2 \to \vec{p}_3\vec{p}_4) f^{(0)}(\vec{p}_1) f^{(0)}(\vec{p}_2)\left[h_i(\vec{p}_1) + h_i(\vec{p}_2)\right]\left[h_i(\vec{p}_3) + h_i(\vec{p}_4) - h_i(\vec{p}_2) - h_i(\vec{p}_1)\right]\\
&= -\frac{1}{4}\int_{\vec{p}_1}\int_{\vec{p}_2}\int_{\vec{p}_3}\int_{\vec{p}_4} \tilde{w}(\vec{p}_1\vec{p}_2 \to \vec{p}_3\vec{p}_4) f^{(0)}(\vec{p}_1) f^{(0)}(\vec{p}_2)\left[h_i(\vec{p}_3) + h_i(\vec{p}_4) - h_i(\vec{p}_2) - h_i(\vec{p}_1)\right]^2 < 0\ , \qquad (43)
\end{aligned}$$

where we exchanged integral variables as $\tilde{w}(\vec{p}_1\vec{p}_2 \to \vec{p}_3\vec{p}_4)$ is symmetric and $f^{(0)}(\vec{p}_1)f^{(0)}(\vec{p}_2) = f^{(0)}(\vec{p}_4)f^{(0)}(\vec{p}_3)$.

v) Based on the assumption that the eigenfunctions $h_i(\vec{p}_1)$ form a complete set, discuss the qualitative properties of the general solution $h(t, \vec{p}_1)$ for a homogenous perturbation in the long time limit $t \to \infty$.

- Because the eigenfunctions form a complete set a general solution can be decomposed into this set

$$h(t, \vec{p}) = \sum_i c_i h_i(\vec{p})\ . \tag{44}$$

The evolution equation is written

$$\begin{aligned}
\partial_t h(t, \vec{p}) &= \sum_i c_i \partial_t h_i(\vec{p})\ , \qquad (45)\\
&= \sum_i c_i \lambda_i h_i(\vec{p}_1)\ . \qquad (46)
\end{aligned}$$

The solution is written then

$$h(t, \vec{p}) = \sum_i c_i e^{\lambda_i t} h_i(\vec{p})\ , \tag{47}$$

as $\lambda_i < 0$, in the long time limit $t \to \infty$ all $e^{\lambda_i t}$ for $i > 1$ deviation will decay exponentially, and we have

$$h(t \to \infty, \vec{p}) = c_n h_n(\vec{p}) + c_\epsilon h_\epsilon(\vec{p}) + \vec{c}_p h_{\vec{p}}(\vec{p})\ , \tag{48}$$

3.2 Entropy conservation in classical dynamics

i) Starting from the Liouville equation for the evolution of the phase-space density $f(t,\{\vec{r}_i\},\{\vec{p}_i\})$ of a classical systems of N particles with two-body interactions

$$\left[\frac{\partial}{\partial t}+\sum_{i=1}^{N}\frac{\vec{p}_i}{m}\vec{\nabla}_{\vec{r}_i}+\sum_{i=1}^{N}\frac{\vec{F}(\vec{r}_i)}{m}\vec{\nabla}_{\vec{p}_i}+\sum_{i=1}^{N}\sum_{j\neq i}\vec{K}_{ij}(|\vec{r}_i-\vec{r}_j|)\vec{\nabla}_{\vec{p}_i}\right]f(t,\{\vec{r}_i\},\{\vec{p}_i\})=0$$

show that the classical entropy

$$S_{\rm cl}(t)=-k_B\int d^{6N}\mathcal{V}\; f(t,\{\vec{r}_i\},\{\vec{p}_i\})\;\log\Big(f(t,\{\vec{r}_i\},\{\vec{p}_i\})\Big)$$

is conserved under Hamiltonian time evolution.
(Hint: What is time derivative of the classical entropy $S_{\rm cl}(t)$?)

- The time derivative of the classical entropy $S_{\rm cl}(t)$ is given by

$$\begin{aligned}\frac{\partial}{\partial t}S_{\rm cl}(t) &= -k_B\int d^{6N}\mathcal{V}\;\Big(1+\log\Big(f(t,\{\vec{r}_i\},\{\vec{p}_i\})\Big)\Big)\,\frac{\partial}{\partial t}f(t,\{\vec{r}_i\},\{\vec{p}_i\})\;, &(49)\\ &= k_B\int d^{6N}\mathcal{V}\;\Big(1+\log\Big(f(t,\{\vec{r}_i\},\{\vec{p}_i\})\Big)\Big)\\ &\qquad\left[\sum_{i=1}^{N}\frac{\vec{p}_i}{m}\vec{\nabla}_{\vec{r}_i}+\sum_{i=1}^{N}\frac{\vec{F}(\vec{r}_i)}{m}\vec{\nabla}_{\vec{p}_i}+\sum_{i=1}^{N}\sum_{j\neq i}\vec{K}_{ij}(|\vec{r}_i-\vec{r}_j|)\vec{\nabla}_{\vec{p}_i}\right]f(t,\{\vec{r}_i\},\{\vec{p}_i\})\;, &(50)\\ &= k_B\int d^{6N}\mathcal{V}\;\left[\sum_{i=1}^{N}\frac{\vec{p}_i}{m}\vec{\nabla}_{\vec{r}_i}+\sum_{i=1}^{N}\frac{\vec{F}(\vec{r}_i)}{m}\vec{\nabla}_{\vec{p}_i}\right.\\ &\qquad\left.+\sum_{i=1}^{N}\sum_{j\neq i}\vec{K}_{ij}(|\vec{r}_i-\vec{r}_j|)\vec{\nabla}_{\vec{p}_i}\right]f(t,\{\vec{r}_i\},\{\vec{p}_i\})\log\Big(f(t,\{\vec{r}_i\},\{\vec{p}_i\})\Big)\;, &(51)\\ &= 0\;, &(52)\end{aligned}$$

because the distribution vanishes at the boundaries $\vec{r}\to\infty$ and $\vec{p}\to\infty$.