Non-equilibrium physics WS 20/21 – Exercise Sheet 6:

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1 Discussion:

i) What is a *mean-field approximation* and under what conditions is it applicable to a systems of interacting particles? What is the physical meaning of the different terms in the Boltzmann equation? What are the assumptions underlying the derivation of the Boltzmann equations?

- *Mean-field approximation* describes the system by averaging over microscopic degrees of freedom in to a field.

- For a system with many degrees of freedom contributing to long-range interactions where correlations can be neglected, the effect of many particles is perceived by one particle as a mean field.

- The Boltzmann equation is given by

$$\left(\partial_t + \underbrace{\vec{v_1}\cdot\vec{\nabla}_{r_1}}_{\text{free streaming}} + \underbrace{\vec{F}\vec{\nabla}_{p_1}}_{\text{external forces}}\right) f_1(t,\vec{r}_1,\vec{p}_1) \;=\; \underbrace{C[f_1](t,\vec{r}_1,\vec{p}_1)}_{\text{Collision terms}} \,. \tag{1}$$

- Boltzmann equation assumptions are
  - Dilute gas : Typical range of interaction much smaller than the mean free path.
  - Molecular chaos : Particles are assumed to be uncorrelated before each scattering follows from stosszahlansatz.
  - Coarse-graining : The evolution timescale is much larger than the collision time and the path length is much larger than interaction length.

2 In-class problems:

2.1 Free-streaming

Consider a *free-streaming* system of particles, which is described by the collisionless Boltzmann equation in the absence of external forces.

i) Show that the general solution for the single particle distribution of a free-streaming system is of the form $f(t,\vec{r},\vec{p}) = f\left(t_0, \vec{r} - \frac{\vec{p}}{m}(t-t_0), \vec{p}\right)$

- The collisionless Boltzmann equation in the absence of external forces is given by

$$(\partial_t + \vec{v}\cdot\vec{\nabla}_r) f(t,\vec{r},\vec{p}) = 0 \,. \tag{2}$$

- We solve this equation using the method of characteristics: We take $\vec{r}(t)$ as a curve trajectory in the space $(\vec{r}, t)$, the full time derivative of the phase-space distribution is given by

$$\frac{df(t,\vec{r},\vec{p})}{dt} = \partial_t f(t,\vec{r},\vec{p}) + (\partial_t \vec{r})\vec{\nabla}_r f(t,\vec{r},\vec{p}) \,, \tag{3}$$

which is fully equivalent to the collisionless Boltzmann equation when we take

$$\partial_t \vec{r} = \vec{v} = \frac{\vec{p}}{m} \,. \tag{4}$$

Meaning, that the collisionless Boltzmann equation is the ordinary differential equation

$$\frac{df(t,\vec{r},\vec{p})}{dt} = 0 \,, \tag{5}$$

taken along the curves $\vec{r}(t)$ such that $\partial_t \vec{r}(t) = \frac{\vec{p}}{m}$.

$f(t,\vec{r},\vec{p})$ is constant along the curve $\vec{r}(t) = \frac{\vec{p}}{m}(t-t_0) + \vec{r}_0$, where $\vec{r}_0$ is the value of the curve at $t = t_0$. The formal solution is then written

$$f(t,\vec{r},\vec{p}) = f(t_0,\vec{r}_0,\vec{p}) = f\left(t_0, \vec{r} - \frac{\vec{p}}{m}(t-t_0), \vec{p}\right) \,. \tag{6}$$

- We can double check by plugging this into the collisionless Bolztmann equation

$$\begin{aligned}(\partial_t + \vec{v}\cdot\vec{\nabla}_r) f\left(t_0, \vec{r} - \frac{\vec{p}}{m}(t-t_0), \vec{p}\right) &= \left(-\partial_t \frac{\vec{p}}{m} t \vec{\nabla}_r + \vec{v}\cdot\vec{\nabla}_r\right) f\left(t_0, \vec{r} - \frac{\vec{p}}{m}(t-t_0), \vec{p}\right) \,, \\ &= 0 \,,\end{aligned} \tag{7}$$

where we used $\partial_t f\left(t_0, \vec{r} - \frac{\vec{p}}{m}(t-t_0), \vec{p}\right) = (\partial_t \frac{\vec{p}}{m}(t-t_0))\vec{\nabla}_{\frac{\vec{p}}{m}(t-t_0)} f\left(t_0, \vec{r} - \frac{\vec{p}}{m}(t-t_0), \vec{p}\right)$, $\vec{\nabla}_r f\left(t_0, \vec{r} - \frac{\vec{p}}{m}(t-t_0), \vec{p}\right) = -\vec{\nabla}_{\frac{\vec{p}}{m}(t-t_0)} f\left(t_0, \vec{r} - \frac{\vec{p}}{m}(t-t_0), \vec{p}\right)$ and $\vec{v} = \frac{\vec{p}}{m}$.

2.2 Constant force

Consider a system of non-interacting particles, which is described by the collisionless Boltzmann equation in the presence of a constant external force $\vec{F}(t,\vec{r}) = \vec{F}_0$.

i) Determine the general solution for the single particle distribution $f(t,\vec{r},\vec{p})$ for a general initial condition $f(t_0,\vec{r},\vec{p}) = f_0(\vec{r},\vec{p})$.

- We solve analogously to the last exercises starting from the collisionless Boltzmann equation in the absence of external forces is given by

$$(\partial_t + \vec{v}\cdot\vec{\nabla}_r + \vec{F}_0\cdot\vec{\nabla}_p) f(t,\vec{r},\vec{p}) = 0 \,. \tag{8}$$

- We solve this equation using the method of characteristics: We take $\vec{r}(t)$ and $\vec{p}(t)$ as curves trajectories in the phase-space $(\vec{r}, \vec{p}, t)$, the full time derivative of the phase-space distribution is given by

$$\frac{df(t,\vec{r},\vec{p})}{dt} = \partial_t f(t,\vec{r},\vec{p}) + (\partial_t \vec{r})\cdot\vec{\nabla}_r f(t,\vec{r},\vec{p}) + (\partial_t \vec{p})\cdot\vec{\nabla}_p f(t,\vec{r},\vec{p}) \,, \tag{9}$$

which is fully equivalent to the collisionless Boltzmann equation when we take

$$\partial_t \vec{r} = \vec{v} = \frac{\vec{p}}{m} \,, \qquad \partial_t \vec{p} = \vec{F}_0 \,. \tag{10}$$

Meaning, that the collisionless Boltzmann equation is the ordinary differential equation

$$\frac{df(t,\vec{r},\vec{p})}{dt} = 0 \,, \tag{11}$$

taken along the curves $(\vec{r}(t), \vec{p}(t))$ such that $\partial_t \vec{r}(t) = \frac{\vec{p}}{m}$ and $\partial_t \vec{p} = \vec{F}_0$.

$f(t,\vec{r},\vec{p})$ is constant along the curve $\vec{r}(t) = \frac{\vec{p}_0}{m}(t-t_0) + \frac{F_0}{2m}(t-t_0)^2 + \vec{r}_0$, $\vec{p}(t) = F_0(t-t_0) + \vec{p}_0$ where $(\vec{r}_0, \vec{p}_0)$ is the value of the curve at $t = t_0$. The formal solution is then written

$$f(t,\vec{r},\vec{p}) = f(t_0,\vec{r}_0,\vec{p}_0) = f\left(t_0, \vec{r} - \frac{\vec{p}_0}{m}(t-t_0) - \frac{F_0}{2m}(t-t_0)^2, \vec{p} - F_0(t-t_0)\right) \,. \tag{12}$$

- We can double check by plugging this into the collisionless Bolztmann equation

$$(\partial_t + \vec{v}\cdot\vec{\nabla}_r + \vec{F}_0\cdot\vec{\nabla}_p) f\left(t_0, \vec{r} - \frac{\vec{p}_0}{m}(t-t_0) - \frac{F_0}{2m}(t-t_0)^2, \vec{p} - F_0(t-t_0)\right) = \tag{13}$$

$$\left(-\partial_t\left(\frac{\vec{p}_0}{m}t + \frac{F_0}{2m}(t-t_0)^2\right)\vec{\nabla}_r - \partial_t\left(F_0 t\right)\vec{\nabla}_p + \vec{v}\cdot\vec{\nabla}_r + \vec{F}_0\cdot\vec{\nabla}_p\right) f(\cdots) \,,$$

$$= 0 \,, \tag{14}$$

where we used $\partial_t f\left(t_0, \vec{r} - X, \vec{p}_0\right) = (\partial_t X)\vec{\nabla}_X f\left(t_0, \vec{r} - X, \vec{p}\right)$,
$\vec{\nabla}_r f\left(t_0, \vec{r} - X, \vec{p}_0\right) = -\vec{\nabla}_X f\left(t_0, \vec{r} - X, \vec{p}\right)$ and $\vec{v} = \frac{\vec{p}}{m} = \frac{\vec{p}_0}{m} + \frac{F_0}{m}(t-t_0)$.

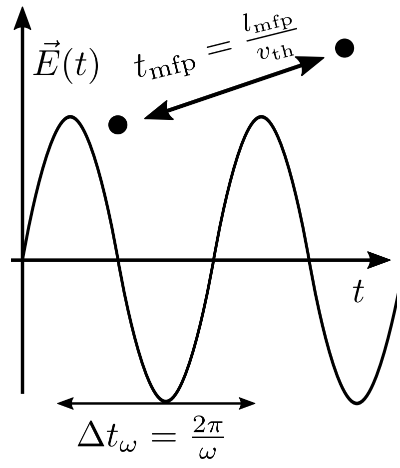

Fig. 1: Electric field representation in time.

3 Homework problems:

3.1 AC conductivity of a collisionless plasma

We consider a plasma of a single species of charged particles, described by the collisionless Boltzmann equation in position ($\vec{r}$) and velocity ($\vec{v} = \frac{\vec{p}_{\rm kin}}{m}$) space

$$\left[\frac{\partial}{\partial t} + \vec{v}\cdot\vec{\nabla}_{\vec{r}} + \frac{e}{m}\left(\vec{E}(t,\vec{r}) + \frac{\vec{v}}{c}\times\vec{B}(t,\vec{r})\right)\vec{\nabla}_{\vec{v}}\right] f(t,\vec{r},\vec{v}) = 0\ , \tag{15}$$

in the presence of a small external electric field

$$\vec{E}(t,\vec{r}) = \lambda \vec{E}_0 e^{i[\vec{k}\vec{r}-\omega t]}\ .$$

and no magnetic field $\vec{B}(t,\vec{r})$.

i) Develop a microscopic picture of the motion of the charged particles over a time scale $\Delta t_\omega = \frac{2\pi}{\omega}$. Discuss how the collisionless approximation can be justified in the limit $v_{\rm th}\Delta t_\omega \ll l_{\rm mfp}$ and why the approximation is not suitable to compute the response to a static electric field ($\omega = 0$).

- The electric field is periodic in time with period Δt_ω, i.e. $\vec{E}(t,\vec{r}) = \vec{E}(t+\Delta t_\omega,\vec{r})$. If the time between collisions $t_{\rm mfp} = \frac{l_{\rm mfp}}{v_{\rm th}}$ is much larger than the periodicity of the electric field, the particles acceleration will oscillate between the electric field peaks and the thermal velocity $v_{\rm th}$ will be periodically modified with no net gain.

  The particles are driven to equilibrium by the field; and we can consider the evolution to be collisionless. For $\omega = 0$ the particles are accelerated to every larger velocities and are free to collide.

Since the general solution to this problem is hard to find, we will instead construct the solution perturbatively, by expanding the single particle distribution according to

$$f(t,\vec{r},\vec{v}) = f_0(\vec{v}) + \lambda\delta f_{\vec{k},\omega}(t,\vec{r},\vec{v})$$

where $\delta f_{\vec{k},\omega}(t,\vec{r},\vec{v}) \ll f_0(\vec{v})$ is a small perturbation generated in response to the external electric field. We consider as our expansion point, an equilibrium distribution of the form

$$f_0(\vec{v}) = n_0 e^{\frac{-m\vec{v}^2}{2k_BT}}\left(\frac{2\pi\hbar^2}{mk_BT}\right)^{3/2}\ ,$$

where n_0 denotes the local particle number density.

ii) Show that the background distribution f_0 satisfies the evolution equation $\left[\frac{\partial}{\partial t} + \vec{v}\cdot\vec{\nabla}_{\vec{r}}\right] f_0 = 0$. Construct the evolution equation for the perturbations $\delta f_{\vec{k},\omega}(t,\vec{r},\vec{v})$ by collecting all residual terms of $O(\lambda)$ in Eq. (15), neglecting terms of order $O(\lambda^2)$ and higher.

- Since the equilibrium distribution does not depend explicitly on time nor space $\vec{r}$ it is a solution to the PDE $\left[\frac{\partial}{\partial t} + \vec{v}\cdot\vec{\nabla}_{\vec{r}}\right] f_0 = 0$.
- The first order evolution equation are written

$$\left[\frac{\partial}{\partial t} + \vec{v}\cdot\vec{\nabla}_{\vec{r}} + \frac{e}{m}\left(\vec{E}(t,\vec{r})\right)\vec{\nabla}_{\vec{v}}\right]\left(f_0(\vec{v}) \quad + \quad \lambda\delta f_{\vec{k},\omega}(t,\vec{r},\vec{v})\right) = 0\,, \tag{16}$$

$$\left[\frac{\partial}{\partial t} + \vec{v}\cdot\vec{\nabla}_{\vec{r}}\right]\lambda\delta f_{\vec{k},\omega}(t,\vec{r},\vec{v}) = -\frac{e}{m}\left(\lambda\vec{E}_0 e^{i[\vec{k}\vec{r}-\omega t]}\right)\vec{\nabla}_{\vec{v}} f_0(\vec{v})\,, \tag{17}$$

$$\left[\frac{\partial}{\partial t} + \vec{v}\cdot\vec{\nabla}_{\vec{r}}\right]\lambda\delta f_{\vec{k},\omega}(t,\vec{r},\vec{v}) = \frac{e\lambda e^{i[\vec{k}\vec{r}-\omega t]}}{k_B T}\left(\vec{v}\cdot\vec{E}_0\right) f_0(\vec{v})\,, \tag{18}$$

Since in the long time limit, the space-time dependence of the perturbation is expected to follow that of the external electric field, we will search for solutions of the form

$$\delta f_{\vec{k},\omega}(t,\vec{r},\vec{v}) = \delta f_{\vec{k},\omega}(\vec{v}) e^{i[\vec{k}\vec{r}-(\omega+i\epsilon)t]}\,,$$

where in the above expression $\epsilon > 0$ is inserted to ensure that $\delta f_{\vec{k},\omega}(\vec{v}) \to 0$ in the limit $t \to -\infty$, and we will take the limit $\epsilon \to 0$ in the final step of our calculation.

iii) Show that the solution for $\delta f_{\vec{k},\omega}(\vec{v})$ can be expressed as

$$\delta f_{\vec{k},\omega}(\vec{v}) = \frac{ie}{k_B T}\frac{\vec{E}_0\cdot\vec{v}}{\omega + i\epsilon - \vec{v}\cdot\vec{k}} f_0(\vec{v}) \tag{19}$$

- The solution can be expressed as

$$\left[\frac{\partial}{\partial t} + \vec{v}\cdot\vec{\nabla}_{\vec{r}}\right]\lambda\left(\delta f_{\vec{k},\omega}(\vec{v}) e^{i[\vec{k}\vec{r}-(\omega+i\epsilon)t]}\right) = \frac{e}{k_B T}\left(\vec{v}\cdot\lambda\vec{E}_0 e^{i[\vec{k}\vec{r}-\omega t]}\right) f_0(\vec{v})\,, \tag{20}$$

$$-i\left[(\omega+i\epsilon) - \vec{v}\cdot\vec{k}\right]\left(\delta f_{\vec{k},\omega}(\vec{v}) e^{-\epsilon t}\right) = \frac{e}{k_B T}\left(\vec{v}\cdot\vec{E}_0\right) f_0(\vec{v})\,, \tag{21}$$

$$\delta f_{\vec{k},\omega}(\vec{v}) = \frac{ie}{k_B T}\frac{\vec{E}_0\cdot\vec{v}}{\omega + i\epsilon - \vec{v}\cdot\vec{k}} f_0(\vec{v})\,. \tag{22}$$

Based on the solution in Eq. (19) we will now proceed to calculate the conductivity tensor $\sigma(\vec{k},\omega)$, which according to $J^i = \sigma^{ij} E^j$ relates the induced current

$$J^i = e\int \frac{m^3 d^3\vec{v}}{(2\pi\hbar)^3}\, v^i\, f(t,\vec{r},\vec{v})$$

to the external electric field $\vec{E}$.

iv) Show the components of the conductivity can be expressed as

$$\sigma^{ij} = \frac{ie^2 m^3}{k_B T}\int \frac{d^3\vec{v}}{(2\pi\hbar)^3}\frac{\vec{v}^i\ \vec{v}^j}{\omega + i\epsilon - \vec{v}\cdot\vec{k}}\, f_0(\vec{v})$$

- The induced current is given by

$$J^i = e\int \frac{m^3 d^3\vec{v}}{(2\pi\hbar)^3}\, v^i\, f(t,\vec{r},\vec{v})\;, \tag{23}$$

$$= e\int \frac{m^3 d^3\vec{v}}{(2\pi\hbar)^3}\, v^i \left(1 + \frac{i\lambda e}{k_B T}\frac{\vec{E}_0\cdot\vec{v}}{\omega + i\epsilon - \vec{v}\cdot\vec{k}} e^{i[\vec{k}\vec{r}-(\omega+i\epsilon)t]}\right) f_0(\vec{v})\;. \tag{24}$$

The first term in the integral is odd in the vector $\vec{v}$ which cancels due to the integral and we have

$$J^i = \frac{ie^2}{k_B T}\int \frac{m^3 d^3\vec{v}}{(2\pi\hbar)^3}\, v^i\, \frac{\vec{E}(t,\vec{r})\cdot\vec{v}}{\omega + i\epsilon - \vec{v}\cdot\vec{k}} f_0(\vec{v})\;, \tag{25}$$

$$\tag{26}$$

- The conductivity can be expressed as

$$\sigma^{ij} = \frac{\partial J^i}{\partial E_j}\;, \tag{27}$$

$$= \frac{ie^2}{k_B T}\int \frac{m^3 d^3\vec{v}}{(2\pi\hbar)^3}\, \frac{v^i v^j}{\omega + i\epsilon - \vec{v}\cdot\vec{k}} f_0(\vec{v})\;. \tag{28}$$

v) Show that the conductivity tensor σ is of the diagonal form $\sigma = diag(\sigma_T, \sigma_T, \sigma_L)$ where $\sigma_{T/L}$ denote the transverse (T) and longitudinal components (L) w.r.t. the wave-vector $\vec{k}$.

- Let us take the z-axis along the k direction, we write

$$\vec{v} = \begin{pmatrix} v_x \\ v_y \\ v_z \end{pmatrix} \qquad \vec{k} = k\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \tag{29}$$

$$\sigma^{ij} = \frac{ie^2}{k_B T}\int \frac{m^3 d^3\vec{v}}{(2\pi\hbar)^3}\, \frac{\vec{v}\otimes\vec{v}}{\omega + i\epsilon - v_z k} f_0(v_x^2, v_y^2, v_z^2)\;. \tag{30}$$

The non-diagonal terms of the conductivity tensor are odd function of $v_{x/y}$ which leads to a vanishing integration from $-\infty$ to ∞, while the diagonal terms are even functions of the $v_{x/y}$. The conductivity tensor is symmetric in exchange of x and y leading to the same transverse σ_T value in these directions while a different longitudinal one in z direction.

vi) Calculate the real and imaginary part of the transverse conductivity $\sigma_T(\vec{k}, \omega)$

- The transverse conductivity is written

$$\sigma_T(\vec{k},\omega) = n_0\left(\frac{2\pi\hbar^2}{mk_B T}\right)^{3/2} \frac{ie^2}{k_B T}\int \frac{m^3 d^3\vec{v}}{(2\pi\hbar)^3}\, \frac{v_x^2}{\omega + i\epsilon - v_z k} e^{\frac{-m\vec{v}^2}{2k_B T}}\;, \tag{31}$$

$$= n_0 \frac{m^3}{(2\pi)^3}\left(\frac{2\pi}{mk_B T}\right)^{3/2} \frac{ie^2}{k_B T}\int dv_z\, \frac{1}{\omega + i\epsilon - v_z k} e^{\frac{-mv_y^2}{2k_B T}} \int dv_x\, v_x^2 e^{\frac{-mv_x^2}{2k_B T}} \int dv_y\, e^{\frac{-mv_y^2}{2k_B T}}\;,$$

$$= n_0 \frac{mk_B T}{(2\pi)^2}\left(\frac{2\pi}{mk_B T}\right)^{3/2} \frac{ie^2}{k}\int dX\, \frac{1}{i\epsilon + X} e^{\frac{-m\left(\frac{\omega - X}{k}\right)^2}{2k_B T}}\;, \tag{32}$$

where $X = \omega - v_z k$ and $\frac{dX}{k} = dv_z$.

- The real part is given by

$$\mathrm{Re}\{\sigma_T(\vec{k},\omega)\} = \frac{n_0 e^2}{2k}\left(\frac{2\pi}{mk_BT}\right)^{1/2} e^{-\frac{m\omega^2}{2k^2k_BT}} . \tag{33}$$

- The imaginary part is given by

$$\mathrm{Im}\{\sigma_T(\vec{k},\omega)\} = \frac{n_0 e^2}{2\pi k}\left(\frac{2\pi}{mk_BT}\right)^{1/2}\int_0^\infty dX\,\frac{e^{\frac{-m(\omega-X)^2}{2k^2k_BT}} - e^{\frac{-m(\omega+X)^2}{2k^2k_BT}}}{X} , \tag{34}$$

$$= \frac{n_0 e^2}{2\pi k}\left(\frac{2\pi}{mk_BT}\right)^{1/2} e^{\frac{-m\omega^2}{2k^2k_BT}}\int_0^\infty dX\,\frac{2\sinh\left(m\frac{X\omega}{k^2k_BT}\right)}{Xk^2} e^{\frac{-mX^2}{2k^2k_BT}} , \tag{35}$$

$$= \frac{n_0 e^2}{2k}\left(\frac{2\pi}{mk_BT}\right)^{1/2} e^{\frac{-m\omega^2}{2k^2k_BT}} Erfi\left(\sqrt{\frac{m}{2k^2k_BT}}\omega\right) , \tag{36}$$

vii) Calculate the real and imaginary part of the longitudinal conductivity $\sigma_L(\vec{k},\omega)$

- The longitudinal conductivity is written

$$\sigma_L(\vec{k},\omega) = n_0\left(\frac{2\pi\hbar^2}{mk_BT}\right)^{3/2}\frac{ie^2}{k_BT}\int\frac{m^3d^3\vec{v}}{(2\pi\hbar)^3}\,\frac{v_z^2}{\omega+i\epsilon-v_zk}e^{\frac{-m\vec{v}^2}{2k_BT}} , \tag{37}$$

$$= n_0\frac{m^3}{(2\pi)^3}\left(\frac{2\pi}{mk_BT}\right)^{3/2}\frac{ie^2}{k_BT}\int dv_z\,\frac{v_z^2}{\omega+i\epsilon-v_zk}e^{\frac{-mv_y^2}{2k_BT}}\left(\int dv_x\,e^{\frac{-mv_x^2}{2k_BT}}\right)^2 ,$$

$$= n_0\frac{m^2}{(2\pi)^2}\left(\frac{2\pi}{mk_BT}\right)^{3/2}\frac{ie^2}{k^3}\int dX\,\frac{(\omega-X)^2}{i\epsilon+X}e^{\frac{-m(\omega-X)^2}{2k^2k_BT}} , \tag{38}$$

- The real part is given by

$$\mathrm{Re}\{\sigma_L(\vec{k},\omega)\} = n_0\omega^2e^2\frac{m^2}{(2\pi)^2k^3}\left(\frac{2\pi}{mk_BT}\right)^{3/2} e^{\frac{m\omega^2}{2k^2k_BT}} . \tag{39}$$

- The imaginary part is given by

$$\mathrm{Im}\{\sigma_L(\vec{k},\omega)\} = n_0e^2\frac{\omega}{2k^3}\left(\left(\frac{2\pi m}{(k_BT)^3}\right)^{1/2} e^{\frac{-m\omega^2}{2k_BT}} Erfi\left(\sqrt{\frac{m}{2k^2k_BT}}\omega\right) - \frac{2k}{k_BT}\right) , \tag{40}$$

(41)

Some hints on the evaluation of integrals

a) Separate the integral $\int d^3\vec{v}$ into integrations over the longitudinal $v_z = \frac{\vec{k}\cdot\vec{v}}{|\vec{k}|}$ and transverse velocities and first perform the integration over the transverse velocities.

b) Express the integral over the longitudinal velocity v_z in the form

$$\lim_{\epsilon\to 0}\int_{-\infty}^{\infty} dx\,\frac{f(x)}{x+i\epsilon}$$

which can be evaluated as

$$\lim_{\epsilon\to 0}\int_{-\infty}^{\infty} dx\ \frac{f(x)}{x+i\epsilon} = -i\pi f(0) + \mathrm{P}\int_{-\infty}^{\infty} dx\ \frac{f(x)}{x}$$

where the principle value (P) of the integral can be calculated according to

$$\mathrm{P}\int_{-\infty}^{\infty} dx\ \frac{f(x)}{x} = \int_{0}^{\infty} dx\ \frac{f(x)-f(-x)}{x}\ .$$

c) Some useful formulae for the remaining integrals include ($\sigma > 0$)

$$\begin{aligned}
\frac{1}{\sqrt{2\pi\sigma^2}}\int_0^{\infty} dx\ \left(\frac{x}{\sigma}\right)\ \sinh(ax/\sigma^2)\ e^{-\frac{x^2}{2\sigma^2}} &= \frac{a}{2\sigma}e^{\frac{a^2}{2\sigma^2}}\ , \\
\frac{1}{\sqrt{2\pi\sigma^2}}\int_0^{\infty} dx\ \frac{\sinh(ax/\sigma^2)}{x/\sigma}\ e^{-\frac{x^2}{2\sigma^2}} &= \frac{1}{2}\sqrt{\frac{\pi}{2}}Erfi\left(\frac{a}{\sqrt{2}\sigma}\right)\ , \\
\frac{1}{\sqrt{2\pi\sigma^2}}\int_0^{\infty} dx\ \cosh(ax/\sigma^2)\ e^{-\frac{x^2}{2\sigma^2}} &= \frac{1}{2}e^{\frac{a^2}{2\sigma^2}}\ ,
\end{aligned} \tag{42}$$