Non-equilibrium physics WS 20/21 – Exercise Sheet 3:

Universität Bielefeld
Instructors: Jun.-Prof. Dr. S. Schlichting, I. Soudi

1 Discussion:

i) What is the physical relevancee of the different terms in the *Navier Stokes equation*?

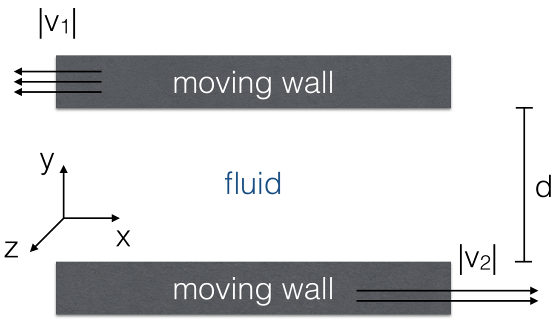
$$\rho \left[\frac{\partial \vec{v}}{\partial t} + (\vec{v} \cdot \vec{\nabla})\vec{v} \right] = -\vec{\nabla} P + \zeta \vec{\nabla} \left(\vec{\nabla} \cdot \vec{v} \right) + \eta \left(\triangle \vec{v} + \frac{1}{3} \vec{\nabla} \left(\vec{\nabla} \cdot \vec{v} \right) \right) \tag{1}$$

- $\frac{\partial \vec{v}}{\partial t}$: Acceleration of the fluid.
- $\frac{\partial \vec{v}}{\partial t} + (\vec{v} \cdot \vec{\nabla})\vec{v}$: Convective derivative.
- $\vec{\nabla} P$: Conservative force, due to gradient of pressure in the fluid.
- ζ : Bulk viscosity, representing the dissipation due to friction in the fluid.
- η : Shear viscosity, representing the dissipation due to shear-flow.

2 In-class problems:

2.1 Shear-flow between two parallel moving plates

Consider an *incompressible* fluid, described by the Navier-Stokes equation with constant transport coefficients ζ, η. The fluid is confined between two parallel plates of infinite extent, separated by a distance d in the y direction and moving with different velocities velocities $\vec{v}_1 = (-|v_1|, 0, 0)$ and $\vec{v}_2 = (|v_2|, 0, 0)$ in the $\pm x$-direction.



i) Write down the equations of motion for the non-vanishing components of the fluid velocity $\vec{v}$.

- Let us take the origin of the coordinate system at the center between the plates. The velocity of the fluid at each boundary follows the plates i.e. $\vec{v}(x = d/2) = (|v_1|, 0, 0)$ and $\vec{v}(x = -d/2) = (|v_2|, 0, 0)$

- Due to the symmetries of the problem, the velocity of the fluid is along the x axis and is only a function of y, i.e. $\vec{v} = (|v|(y), 0, 0)$. We have then

$$\vec{\nabla}.\vec{v} = 0 \; , \qquad (\vec{v}.\vec{\nabla})\vec{v} = 0 \; , \qquad \Delta\vec{v} = (\frac{\partial^2}{\partial y^2} v_x, 0, 0) \; . \tag{2}$$

Leading to the simplified Navier-Stokes equations

$$\rho \frac{\partial \vec{v}}{\partial t} = \eta \triangle \vec{v} \Rightarrow \rho \frac{\partial v_x}{\partial t} = \eta \frac{\partial^2 v_x}{\partial y^2} \tag{3}$$

ii) Determine the stationary velocity profile $\vec{v}(x, y, z)$ of the fluid.

- The stationary profile is determined from the solution of Eq. (3) for stationary velocity

$$\eta \frac{\partial^2 v_x}{\partial y^2} = 0 \quad \Rightarrow \quad v_x(y) = c_1 + c_2 y \; . \tag{4}$$

By using the boundary conditions we can solve for c_1, c_2

$$\begin{aligned} v_x(-d/2) &= |v_2| = c_1 - c_2 d/2 \; , \\ v_x(d/2) &= -|v_1| = c_1 + c_2 d/2 \; , \\ c_1 = \frac{|v_2| - |v_1|}{2} \; , &\qquad c_2 = -\frac{|v_1| + |v_2|}{d} \; . \end{aligned} \tag{5}$$

3 Homework problems:

3.1 Sound & shear waves

Consider a compressible fluid of constant density $\rho(t,\vec{x}) = \rho_0$, with an equation of state $p = c_s^2 \rho$ which is at rest $\vec{v}(t,\vec{x}) = 0$ in the laboratory frame. We will consider the propagation of density $\delta\rho$ and velocity perturbations δv, whose evolution is described by the linearized continuity equation and the linearized Navier-Stokes equation. We first consider so called sound waves, which are perturbations of the form

$$\delta\rho(t,\vec{x}) = \delta\rho_{\vec{k}}(t) e^{i\vec{k}\vec{x}} , \qquad \delta\vec{v}(t,\vec{x}) = \frac{\vec{k}}{|\vec{k}|}\delta v_{\vec{k},\parallel}(t) e^{i\vec{k}\vec{x}}$$

i) Derive the linearized evolution equations for sound waves.

- The linearized continuity equation is given by

$$\begin{aligned} \frac{\partial\rho}{\partial t} + \vec{\nabla}\cdot(\rho\vec{v}) &= 0 \,, & (6)\\ \frac{\partial\delta\rho}{\partial t} + \rho_0\vec{\nabla}\cdot\delta\vec{v} &= 0 \,, & (7)\\ \frac{\partial\delta\rho_{\vec{k}}(t)}{\partial t} + ik\rho_0\delta v_{\vec{k},\parallel}(t) &= 0 \,. & (8) \end{aligned}$$

- The linearized Navier-Stokes equation is given by

$$\begin{aligned} \rho\left[\frac{\partial\vec{v}}{\partial t} + (\vec{v}\cdot\vec{\nabla})\vec{v}\right] &= -\vec{\nabla}P + \zeta\vec{\nabla}\left(\vec{\nabla}\cdot\vec{v}\right) + \eta\left(\triangle\vec{v} + \frac{1}{3}\vec{\nabla}\left(\vec{\nabla}\cdot\vec{v}\right)\right) \,, & (9)\\ \rho_0\frac{\partial\delta\vec{v}}{\partial t} &= -c_s^2\vec{\nabla}\delta\rho + \zeta\vec{\nabla}\left(\vec{\nabla}\cdot\delta\vec{v}\right) + \eta\left(\triangle\delta\vec{v} + \frac{1}{3}\vec{\nabla}\left(\vec{\nabla}\cdot\delta\vec{v}\right)\right) \,, & (10)\\ \rho_0\frac{\vec{k}}{|\vec{k}|}\frac{\partial\delta v_{\vec{k},\parallel}(t)}{\partial t} &= -ic_s^2\,\vec{k}\delta\rho_{\vec{k}}(t) - \zeta\,\vec{k}|\vec{k}|\delta v_{\vec{k},\parallel}(t) - \eta\left(\vec{k}|\vec{k}|\delta v_{\vec{k},\parallel}(t) + \frac{1}{3}\vec{k}|\vec{k}|\delta v_{\vec{k},\parallel}(t)\right) \,, & (11)\\ \rho_0\frac{\vec{k}}{|\vec{k}|}\frac{\partial\delta v_{\vec{k},\parallel}(t)}{\partial t} &= -ic_s^2\,\vec{k}\delta\rho_{\vec{k}}(t) - \vec{k}|\vec{k}|\delta v_{\vec{k},\parallel}(t)\left(\zeta + \frac{4}{3}\eta\right) \,. & (12) \end{aligned}$$

ii) Show that the general solution for sound waves can be expressed in the form

$$\begin{aligned} \delta\rho_{\vec{k}}(t) &= \sum_{\pm} c^{\pm}_{\vec{k}} \;\; \rho_0 \;\; e^{i\omega_{\pm}(\vec{k})t} \,, & (13)\\ \delta v_{\vec{k},\parallel}(t) &= \sum_{\pm} c^{\pm}_{\vec{k}} \left(\frac{-\omega_{\pm}(\vec{k})}{|\vec{k}|}\right) e^{i\omega_{\pm}(\vec{k})t} \,, & (14) \end{aligned}$$

and determine the dispersion relation $\omega_{\pm}(\vec{k})$. What differences do you observe between ideal $(\eta = \zeta = 0)$ and viscous fluid dynamics?

- Taking derivatives of the Navier-Stokes equation

$$\begin{aligned} \rho_0\frac{\vec{k}}{|\vec{k}|}\frac{\partial^2\delta v_{\vec{k},\parallel}(t)}{\partial t^2} &= -ic_s^2\,\vec{k}\frac{\partial\delta\rho_{\vec{k}}(t)}{\partial t} - \vec{k}|\vec{k}|\frac{\partial\delta v_{\vec{k},\parallel}(t)}{\partial t}\left(\zeta + \frac{4}{3}\eta\right) \,, & (15)\\ \rho_0\frac{\partial^2\delta v_{\vec{k},\parallel}(t)}{\partial t^2} &= -c_s^2\,\vec{k}^2\rho_0\delta v_{\vec{k},\parallel}(t) - \vec{k}^2\frac{\partial\delta v_{\vec{k},\parallel}(t)}{\partial t}\left(\zeta + \frac{4}{3}\eta\right) \,, & (16) \end{aligned}$$

where we used the continuity equation to simplify the density. We define

$$\gamma = \frac{1}{\rho_0}\left(\zeta + \frac{4}{3}\eta\right) \ . \tag{17}$$

Solving the characteristic equation $-\omega^2 = -c_s^2\,\vec{k}^2 + i\gamma\vec{k}^2\omega$, taking $\vec{k}^2$ to be real and $\omega = \omega_r + i\omega_i$ complex we have

$$-\omega_r^2 + \omega_i^2 - 2i\omega_r\omega_i = -c_s^2\,\vec{k}^2 + i\gamma\vec{k}^2(\omega_r + i\omega_i) \ , \tag{18}$$
$$-\omega_r^2 + \omega_i^2 - 2i\omega_r\omega_i = -c_s^2\,\vec{k}^2 + i\gamma\vec{k}^2\omega_r + \gamma\vec{k}^2\omega_i \ . \tag{19}$$

We find

$$\omega_\pm(\vec{k}) = \pm\sqrt{c_s^2\vec{k}^2 - \frac{1}{4}\gamma^2\vec{k}^4} - i\frac{\gamma\vec{k}^2}{2} \ . \tag{20}$$

The solution to the differential equation is written

$$\delta v_{\vec{k},\parallel}(t) = \sum_\pm c_{\vec{k}}^\pm \left(\frac{-\omega_\pm(\vec{k})}{|\vec{k}|}\right) e^{i\omega_\pm(\vec{k})t} \ , \tag{21}$$

where we have taken a factor $\left(\frac{-\omega_\pm(\vec{k})}{|\vec{k}|}\right)$ upfront simplifying the differential equation for density

$$\frac{\partial\delta\rho_{\vec{k}}(t)}{\partial t} = -ik\rho_0\delta v_{\vec{k},\parallel}(t) \ , \tag{22}$$
$$\frac{\partial\delta\rho_{\vec{k}}(t)}{\partial t} = -ik\rho_0\sum_\pm c_{\vec{k}}^\pm \left(\frac{-\omega_\pm(\vec{k})}{|\vec{k}|}\right) e^{i\omega_\pm(\vec{k})t} \ , \tag{23}$$
$$\delta\rho_{\vec{k}}(t) = \sum_\pm c_{\vec{k}}^\pm \ \rho_0 \ e^{i\omega_\pm(\vec{k})t} \ . \tag{24}$$

Next we will consider shear waves, which are velocity perturbations of the form

$$\delta\vec{v}(t,\vec{x}) = \delta\vec{v}_{\vec{k},\perp}(t)e^{i\vec{k}\vec{x}} \tag{25}$$

which are transverse to the wave-vector $\vec{k}$, meaning that $\delta\vec{k}\cdot\vec{v}_{\vec{k},\perp}(t) = 0$.

iii) Show with the help of the linearized continuity equation that shear waves do not induce density perturbations, i.e. that they are consistent with $\delta\rho = 0$.

- The linear continuity equation is given by

$$\frac{\partial\rho}{\partial t} + \vec{\nabla}\cdot(\rho\vec{v}) = 0 \ , \tag{26}$$
$$\frac{\partial\delta\rho}{\partial t} + \rho_0\vec{\nabla}\cdot\delta\vec{v} = 0 \ , \tag{27}$$
$$\frac{\partial\delta\rho_{\vec{k}}(t)}{\partial t} = 0 \ . \tag{28}$$

Leading to $\delta\rho_{\vec{k}}(t) = \delta\rho_{\vec{k}}(0) = 0$

iv) Derive the linearized evolution equation for shear waves. Show that the solution can be expressed in the form

$$\delta\vec{v}_{\vec{k},\perp}(t) = \vec{c}^{\,\perp}_{\vec{k}} e^{+i\omega(\vec{k})t} \,, \tag{29}$$

and determine the dispersion relation $\omega(\vec{k})$. What differences do you observe between shear waves and sound waves?

- Firstly, we have the following identities for shear waves velocity pertrubations

$$\vec{\nabla}\cdot\delta\vec{v} = i\vec{k}\cdot\delta\vec{v}_{\vec{k},\perp}(t)e^{i\vec{k}\vec{x}} = 0 \,, \qquad \Delta\delta\vec{v} = -\vec{k}^2\delta\vec{v}_{\vec{k},\perp}(t)e^{i\vec{k}\vec{x}} \,. \tag{30}$$

- The linear Navier-Stokes equation is given by

$$\rho_0\frac{\partial\delta\vec{v}_{\vec{k},\perp}(t)}{\partial t}e^{i\vec{k}\vec{x}} = -\eta\,\vec{k}^2\delta\vec{v}_{\vec{k},\perp}(t)e^{i\vec{k}\vec{x}} \,, \tag{31}$$

$$\frac{\partial\delta\vec{v}_{\vec{k},\perp}(t)}{\partial t} = -\frac{\eta\,\vec{k}^2}{\rho_0}\delta\vec{v}_{\vec{k},\perp}(t) \,. \tag{32}$$

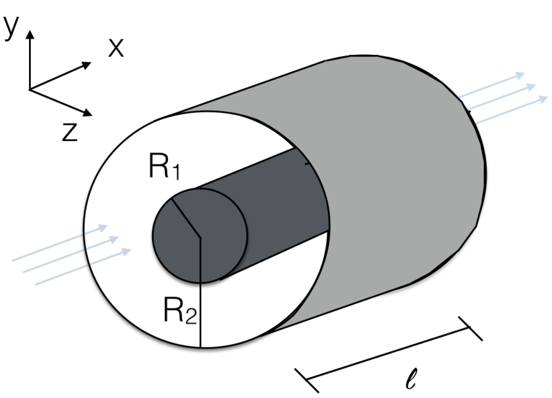
The solution is given by

$$\delta\vec{v}_{\vec{k},\perp}(t) = \vec{c}^{\,\perp}_{\vec{k}} e^{+i\omega(\vec{k})t} \,, \tag{33}$$

where the dispersion relation is $\omega(\vec{k}) = i\frac{\eta\vec{k}^2}{\rho_0}$. The dispersion relation is imaginary for shear waves, leading to an exponentially decaying perturbation. While for sound waves, the perturbation is oscillatory with a damping factor due to viscosity for ($|\vec{k}| \gg 1$ or $t \gg 1$).

3.2 Flow in a donut pipe

Consider the flow of an incompressible fluid in a "donut shaped" pipe of length l, with inner and outer radii R_1 and R_2, as illustrated in the figure.



i) Determine the velocity profile $v_x(r)$ for stationary transport, for a given pressure difference Δp between the two ends of the pipe ($r = \sqrt{y^2+z^2}$).

- The continuity equation is given by

$$\underbrace{\frac{\partial\rho}{\partial t}}_{=0\ \text{(Stationary)}} + \underbrace{\vec{\nabla}\cdot(\rho\vec{v})}_{=\rho\vec{\nabla}\cdot(\vec{v})\ \text{(Incompressible)}} = 0 \,, \tag{34}$$

leading to

$$\frac{\partial v_x(r)}{\partial x} = 0 \,. \tag{35}$$

- The Navier-Stokes equations are written

$$\rho\left[\frac{\partial\vec{v}}{\partial t}+(\vec{v}\cdot\vec{\nabla})\vec{v}\right] = -\vec{\nabla}P+\zeta\vec{\nabla}\left(\vec{\nabla}\cdot\vec{v}\right)+\eta\left(\triangle\vec{v}+\frac{1}{3}\vec{\nabla}\left(\vec{\nabla}\cdot\vec{v}\right)\right) , \tag{36}$$

$$\frac{\partial}{\partial x}P = \eta\triangle v_x(r) , \tag{37}$$

where we cancelled the terms according to Eq. (35) and that the stationary condition.

- Taking η to be constant and evaluating the x derivative we find

$$\frac{\partial^2}{\partial x^2}P = \eta\triangle\frac{\partial}{\partial x}v_x(r)=0 , \tag{38}$$

$$\frac{\partial}{\partial x}P = \text{const}=\frac{P_{\text{out}}-P_{\text{in}}}{l}=-\frac{\Delta P}{l} , \tag{39}$$

Leading to the equation

$$\eta\triangle v_x(r) = -\frac{\Delta P}{l} , \tag{40}$$

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}\right)v_x(r) = -\frac{\Delta P}{\eta l} , \tag{41}$$

where we expressed the Laplacian in cylindrical coordinates. Integrating over r, we find

$$r\frac{\partial}{\partial r}v_x(r) = -\frac{\Delta P}{2\eta l}r^2+c_1 , \tag{42}$$

where c_1 is an integration constant. Integrating a second time we find

$$v_x(r) = -\frac{\Delta P}{4\eta l}r^2+c_1\log(r)+c_2 . \tag{43}$$

- Let us determine $c_{1/2}$, using boundary conditions

$$v_x(R_1)=0 , \qquad v_x(R_2)=0 , \tag{44}$$

$$-\frac{\Delta P}{4\eta l}R_1^2+c_1\log(R_1)+c_2=0 , \qquad -\frac{\Delta P}{4\eta l}R_2^2+c_1\log(R_2)+c_2=0 , \tag{45}$$

$$c_1 = \frac{\Delta P}{4\eta l}\frac{R_1^2-R_2^2}{\log(R_1/R_2)} \tag{46}$$

$$c_2 = \frac{\Delta P}{4\eta l}\frac{R_2^2\log(R_1)-R_1^2\log(R_2)}{\log(R_1/R_2)} \tag{47}$$

- Cross-check with lecture where $R_1=0$

$$c_1(R_1\to 0) = 0 , \tag{48}$$

$$c_2(R_1\to 0) = \frac{\Delta P}{4\eta l}R^2 , \tag{49}$$

ii) Calculate the rate of mass transport (or discharge) $Q=\int d^2\vec{\sigma}\ (\rho\vec{v})$ for the donut pipe.

-

$$
\begin{aligned}
Q &= \int d^2\vec{\sigma}\ (\rho\vec{v})\ , && (50)\\
&= 2\pi\rho\int_{R_1}^{R_2} rdr\ v_x(r)\ , && (51)\\
&= 2\pi\rho\int_{R_1}^{R_2} rdr\ -\frac{\Delta P}{4\eta l}r^2 + c_1\log(r) + c_2\ . && (52)\\
&= \pi\rho\frac{\Delta P\left(\left(R_2^4 - R_1^4\right)\log(R_1/R_2) + \left(R_1^2 - R_2^2\right)^2\right)}{8\eta l(\log(R_1/R_2))} && (53)
\end{aligned}
$$

- Cross-check with lecture where $R_1 = 0$

$$
Q = \pi\rho\frac{\Delta P R_2^4}{8\eta l}\ . \qquad (54)
$$