Non-equilibrium physics WS 20/21 – Exercise Sheet 11:

Universität Bielefeld
Instructors: Jun.-Prof. Dr. S. Schlichting, I.Soudi

1 Discussion:

i) What are the physical differences between stationary turbulent solutions and equilibrium solutions of kinetic equations? What are direct and inverse cascades?

- The stationary turbulent solutions are solution of the kinetic equation describing a dynamical system of a transfer of a conserved quantity injected and dissipated at a steady rate. They satisfy balance principle and have a finite flux.
- While the equilibrium solutions are late time behavior of the kinetic equations without dissipation and describes a stationary system in equilibrium. They satisfy detailed balance and zero flux.
- Direct/Inverse cascades describe the direction of transportation of the conserved quantity. Direct for a cascade from low momentum to higher momentum and Inverse is a cascade in the opposite direction.

2 In-class problems:

2.1 Kolmogorov-Zakharov Spectra

Consider the three wave kinetic equation $\partial_t n(k,t) = I[n](k,t)$ for a statistically homogenous and isotropic system of waves, with power law dispersion $\omega(k) = \omega_0(k/k_0)^z$ with the collision integral given by

$$\begin{aligned} I[n](k,t) \;=\; \frac{1}{2}\int d^dk_1 \int d^dk_2 \Big\{ \;& \tilde{w}(k \to k_1k_2)\Big[n(k_1,t)n(k_2,t) - n(k,t)\big(n(k_1,t)+n(k_2,t)\big)\Big] \\ & - \tilde{w}(k_1 \to k_2k)\Big[n(k_2,t)n(k,t) \; - n(k_1,t)\big(n(k_2,t)+n(k,t)\big)\Big] \\ & - \tilde{w}(k_2 \to kk_1)\Big[n(k,t)n(k_1,t) \; - n(k_2,t)\big(n(k,t)+n(k_1,t)\big)\Big]\Big\} \end{aligned} \quad (1)$$

where $\widetilde{w}(k \to k_1k_2) = (2\pi)\delta(\omega(k)-\omega(k_1)-\omega(k_2))\delta^{(d)}(k-k_1-k_2)|V(k,k_1,k_2)|^2$ for a scale invariant matrix element $V(\lambda k, \lambda k_1, \lambda k_2) = \lambda^m V(k,k_1,k_2)$ with the usual symmetry properties $V(k,k_1,k_2) = V(k,k_2,k_1)$.

i) Show that for a spectrum of the Kolmogorov-Zakharov (KZ) form $n(k,t) = n_{\rm KZ}(k) = n_0\left(\frac{k}{k_0}\right)^{-s_0}$ the energy flux $J_k(\Lambda,t) = -\int_0^\Lambda d^dk\; \omega(k)\; I[n](k,t)$ through a momentum shell can be expressed as

$$J_k(\Lambda) = -\omega_0 k_0^{d-(\xi+z)}\Omega^{(d)}\int_0^\Lambda dk\; k^{\xi+z-1} I[n_{\rm KZ}](k_0)\;,$$

with $\xi = 2d - z + 2m - 2s_0$.

- Using the following Zakharov transformation

$$k = k_0 \frac{k_0}{k'} \;, \qquad k_1 = k_1' \frac{k_0}{k'} \;, \qquad k_2 = k_2' \frac{k_0}{k'} \;, \tag{2}$$

we write the different terms of the integrand

$$\begin{aligned}
dk_1 &= \frac{k_0}{k'} dk_1' \;, \qquad dk_2 = \frac{k_0}{k'} dk_2' \;, & (3)\\
(k_1 k_2)^{2(d-1)} dk_1 dk_2 &= (k_1' k_2')^{2(d-1)} \left(\frac{k_0}{k'}\right)^{2d} dk_1' dk_2' & (4)\\
n(k,t) &= n_0 \left(\frac{k_0}{k'}\right)^{-s_0} \;, \qquad n(k_1,t) = n(k_1',t) \left(\frac{k_0}{k'}\right)^{-s_0} \;, & (5)\\
n(k_2,t) &= n(k_2',t) \left(\frac{k_0}{k'}\right)^{-s_0} \;, & (6)\\
\delta(\omega(k) - \omega(k_1) - \omega(k_2)) &= \delta\omega_0 \left(\left(\frac{k}{k_0}\right)^z - \left(\frac{k_1}{k_0}\right)^z - \left(\frac{k_2}{k_0}\right)^z \right) \;, & (7)\\
&= \delta\omega_0 \left(\left(\frac{k_0}{k'}\right)^z - \left(\frac{k_1' k_0}{k' k_0}\right)^z - \left(\frac{k_0 k_2'}{k' k_0}\right)^z \right) \;, & (8)\\
&= \left(\frac{k_0}{k'}\right)^{-z} \delta(\omega(k_0) - \omega(k_1') - \omega(k_2')) \;, & (9)\\
\delta^{(d)}(k - k_1 - k_2) &= \left(\frac{k_0}{k'}\right)^{-d} \delta(k_0 - k_1' - k_2') \;, & (10)\\
|V(k,k_1,k_2)|^2 &= \left(\frac{k_0}{k'}\right)^{2m} |V(k',k_1',k_2')|^2 \;. & (11)
\end{aligned}$$

- The collision integral becomes

$$\begin{aligned}
I[n_{\mathrm{KZ}}](k,t) &= I[n_{\mathrm{KZ}}](k_0,t) \left(\frac{k_0}{k'}\right)^{2m-d-z-2s_0+2d} \;, & (12)\\
&= I[n_{\mathrm{KZ}}](k_0,t) \left(\frac{k}{k_0}\right)^{2m-d-z-2s_0+2d} \;, & (13)\\
& & (14)
\end{aligned}$$

- Using Kolmogorov-Zakharov spectrum, the energy flux through a momentum shell is written

$$\begin{aligned}
J_k(\Lambda) &= -\int_0^\Lambda d^d k \; \omega(k) \; I[n_{\mathrm{KZ}}](k,t) \;, & (15)\\
&= -\Omega^{(d)} \int_0^\Lambda dk \; k^{d-1} \; \omega_0 (k/k_0)^z \left(\frac{k}{k_0}\right)^{2m-d-z-2s_0+2d} I[n_{\mathrm{KZ}}](k_0,t) \;, & (16)\\
&= -\omega_0 k_0^{d-(\xi+z)} \Omega^{(d)} \int_0^\Lambda dk k^{\xi+z-1} I[n_{\mathrm{KZ}}](k_0,t) \;, & (17)\\
& & (18)
\end{aligned}$$

ii) Determine the scaling exponent s_0 of the KZ spectrum based on the condition that the energy flux $J_k(\Lambda,t) = -\int_0^\Lambda d^d k \; \omega(k) \; I[n](k,t)$ through a momentum shell becomes scale (Λ) invariant. Explain why this condition is necessary for $n(k,t) = n_{\mathrm{KZ}}(k)$ to be a stationary turbulent solution.

- The collision integral $I[n_{\mathrm{KZ}}](k_0,t)$ does not depend on k and can be taken out of the integral and the remaining integral is $\int_0^\Lambda dk\; k^{\xi+z-1}$, which determines the dependence of the energy flux on the scale λ.

- A stationary turbulent solution describe a scale invariant energy flux, that is why we need $J_k(\Lambda)$ to be independent of Λ, giving

$$\frac{dJ_k(\Lambda)}{d\Lambda} \;=\; 0 \propto \frac{d}{d\Lambda}\int_0^\Lambda dk\; k^{\xi+z-1} \propto \frac{d}{d\Lambda}\Lambda^{\xi+z} \tag{19}$$

which means $\xi + z \sim 0 \Rightarrow s_0 = d + m$.

3 Homework problems:

3.1 Self-similar solutions

Consider the decaying turbulence of a statistically homogenous and isotropic system of waves described by the three wave kinetic equation $\partial_t n(k,t) = I[n](k,t)$.

iii) Determine the dynamical scaling exponents α, β of the self-similar solutions

$$n(k,t) = (t/t_0)^\alpha n_S\left(k(t/t_0)^\beta\right) ,$$

of the kinetic equation when a) no energy is injected or removed from the system and b) energy is injected into the system at a constant rate $\dot\epsilon = \dot\epsilon_0$.

- When no energy is injected or removed from the system the total energy is constant

$$E(t) \;=\; \int d^dk\; \omega(k) n(k,t) = \text{const} . \tag{20}$$

Substituting the self-similar solution

$$E(t) \;=\; \int d^dk\; \omega_0 \left(\frac{k}{k_0}\right)^z \left(\frac{t}{t_0}\right)^\alpha n_S\left(k\left(\frac{t}{t_0}\right)^\beta\right) . \tag{21}$$

Setting $x = k\left(\frac{t}{t_0}\right)^\beta$, we have $d^dk = \left(\frac{t}{t_0}\right)^{-d\beta} dx$

$$E(t) \;=\; \omega_0 \left(\frac{1}{k_0}\right)^z \left(\frac{t}{t_0}\right)^{\alpha-(d+z)\beta} \int d^dx\; x^z n_S(x) . \tag{22}$$

Since $\int d^dx\; x^z n_S(x)$ is constant, we have

$$\alpha - (d+z)\beta = 0 . \tag{23}$$

To obtain the second scaling exponent equation, we substitute the self-similar solution in three wave kinetic equation

$$\partial_t n(k,t) \;=\; I[n](k,t) , \tag{24}$$

from the LHS we obtain

$$\partial_t n(k,t) \;=\; \left(\frac{t}{t_0}\right)^{\alpha-1} \frac{1}{t_0}\left[\alpha n_S(x) + \beta x \partial_x n_S(x)\right] . \tag{25}$$

and from the RHS we obtain $\left(\text{define } x_i = k_i \left(\frac{t}{t_0}\right)^{\beta} = k_i Q \Rightarrow k_i = x_i/Q\right)$

$$\begin{aligned} I[n](k,t) &= \frac{1}{2}\int d^dk_1 \int d^dk_2 \Big\{ \tilde{w}(k \to k_1 k_2)\Big[n(k_1,t)n(k_2,t) - n(k,t)\big(n(k_1,t)+n(k_2,t)\big)\Big] + \ldots \quad (26) \\ &= \frac{1}{2}\left(\frac{t}{t_0}\right)^{2\alpha-2d\beta} \int d^dx_1 \int d^dx_2 \Big\{ \tilde{w}(x/Q \to x_1/Q x_2/Q) \\ &\qquad \times \Big[n_S(x_1)n_S(x_2) - n_S(x)\big(n_S(x_1)+n_S(x_2)\big)\Big] + \ldots , \quad (27) \end{aligned}$$

We have

$$\tilde{w}(x/Q \to x_1/Q x_2/Q) = Q^{d+z-2m}\tilde{w}(x \to x_1 x_2) . \quad (28)$$

$$\begin{aligned} I[n](k,t) &= \left(\frac{t}{t_0}\right)^{2\alpha+(-2d+d+z-2m)\beta} I[n_s](x,t) , \quad (29) \\ \left(\frac{t}{t_0}\right)^{\alpha-1}\frac{1}{t_0}\left[\alpha n_S(x) + \beta x \partial_x n_S(x)\right] &= \left(\frac{t}{t_0}\right)^{2\alpha+(-2d+d+z-2m)\beta} I[n_s](x,t) . \quad (30) \\ & \quad (31) \end{aligned}$$

Leading to the set of equations

$$\begin{aligned} \alpha - 1 &= 2\alpha + (z-d-2m)\beta , \quad (32) \\ \alpha &= (d+z)\beta . \quad (33) \end{aligned}$$

Which are solved by

$$\alpha = \frac{d+z}{2(m-z)} , \qquad \beta = \frac{1}{2(m-z)} . \quad (34)$$

- In the case where energy is injected and removed from the system at a constant rate we have

$$E(t) = \int d^dk \, \omega(k)n(k,t) \propto t . \quad (35)$$

The equations become

$$\begin{aligned} \alpha - 1 &= 2\alpha + (z-d-2m)\beta , \quad (36) \\ \alpha &= (d+z)\beta + \underbrace{1}_{E\propto t} . \quad (37) \end{aligned}$$

Which are solved by

$$\alpha = \frac{d+m}{m-z} , \qquad \beta = \frac{1}{m-z} . \quad (38)$$

3.2 Stationary turbulence of capillary waves on shallow water

Consider again the three wave kinetic equation $\partial_t n(k,t) = I[n](k,t)$ for a statistically homogenous and isotropic system of waves.

i) Show that for a spectrum of the Kolmogorov-Zakharov (KZ) form $n(k,t) = n_{\rm KZ}(k) = n_0 \left(\frac{k}{k_0}\right)^{-s_0}$ the collision kernel $I[n](k,t)$ in Eq. (1) evaluated at $k = k_0$ can be expressed as

$$I[n_{\rm KZ}](k_0) = \frac{n_0^2}{2\Omega^{(d)}} \int d\Omega_0 \int d^d k_1 \int d^d k_2 \; \tilde{w}(k_0 \to k_1 k_2)$$
$$\left[\left|\frac{k_0}{k_1}\right|^{s_0} \left|\frac{k_0}{k_2}\right|^{s_0} - \left(\left|\frac{k_0}{k_1}\right|^{s_0} + \left|\frac{k_0}{k_2}\right|^{s_0} \right) \right] \times \left[1 - \left|\frac{k_0}{k_1}\right|^{\xi} - \left|\frac{k_0}{k_2}\right|^{\xi} \right]$$

with $\xi = 2d - z + 2m - 2s_0$. (Hint: Zakharov transformation)

- Starting from the collision integral

$$\begin{aligned} I[n_{\rm KZ}](k_0,t) \;=\; \frac{1}{2} \int d^d k_1 \int d^d k_2 \Big\{ \;& \tilde{w}(k_0 \to k_1 k_2) \Big[n(k_1,t) n(k_2,t) - n(k_0,t) \big(n(k_1,t) + n(k_2,t) \big) \Big] \\ & - \tilde{w}(k_1 \to k_2 k_0) \Big[n(k_2,t) n(k_0,t) \; - n(k_1,t) \big(n(k_2,t) + n(k_0,t) \big) \Big] \\ & - \tilde{w}(k_2 \to k_0 k_1) \Big[n(k_0,t) n(k_1,t) \; - n(k_2,t) \big(n(k_0,t) + n(k_1,t) \big) \Big] \Big\} \end{aligned}$$

- We can rewrite the term $\tilde{w}(k_2 \to k_0 k_1)$,using the following Zakharov transformation

$$k_0 = k_1' \frac{k_0}{k_1'} \;, \qquad k_1 = k_0 \frac{k_0}{k_1'} \;, \qquad k_2 = k_2' \frac{k_0}{k_1'} \;, \tag{39}$$

we write the different terms of the integrand

$$dk_1 \;=\; \frac{k_0^2}{k_1'^2} dk_1' \;, \qquad dk_2 = \frac{k_0}{k_1'} dk_2' \;, \tag{40}$$

$$(k_1 k_2)^{(d-1)} dk_1 dk_2 \;=\; (k_1' k_2')^{(d-1)} \left(\frac{k_0}{k_1'}\right)^{3d} dk_1' dk_2' \tag{41}$$

$$n(k_0,t) \;=\; n(k_1',t) \left(\frac{k_0}{k_1'}\right)^{-s_0} \;, \qquad n(k_1,t) = n_0 \left(\frac{k_0}{k_1'}\right)^{-s_0} \;, \tag{42}$$

$$n(k_2,t) \;=\; n(k_2',t) \left(\frac{k_0}{k_1'}\right)^{-s_0} \;, \tag{43}$$

$$\delta(\omega(k_2) - \omega(k_0) - \omega(k_1)) \;=\; \delta\omega_0 \left(\left(\frac{k}{k_0}\right)^z - \left(\frac{k_1}{k_0}\right)^z - \left(\frac{k_2}{k_0}\right)^z \right) \;, \tag{44}$$

$$\;=\; \left(\frac{k_0}{k_1'}\right)^{-z} \delta(\omega(k_2') - \omega(k_0) - \omega(k_1')) \;, \tag{45}$$

$$\delta^{(d)}(k_2 - k_0 - k_1) \;=\; \left(\frac{k_0}{k_1'}\right)^{-d} \delta(k_2' - k_0 - k_1') \;, \tag{46}$$

$$|V(k,k_1,k_2)|^2 \;=\; \left(\frac{k_0}{k_1'}\right)^{2m} |V(k_2',k_0,k_1')|^2 \;. \tag{47}$$

- The collision integral becomes

$$\begin{aligned} & - \int d^d k_1 d^d k_2 \tilde{w}(k_2 \to k_0 k_1) \Big[n(k_0,t) n(k_1,t) \; - n(k_2,t) \big(n(k_0,t) + n(k_1,t) \big) \Big] \Big\} \\ & = - \int d^d k_1' \int d^d k_2' \tilde{w}(k_2' \to k_0' k_1') \left(\frac{k_0}{k_1'}\right)^{3d+2m-d-z-2s_0} \Big[n(k_0,t) n(k_1',t) \; - n(k_2',t) \big(n(k_0,t) + n(k_1',t) \big) \Big] \Big\} \end{aligned} \tag{48}$$

Using analogous transformation of the other terms we can write

$$I[n_{\mathrm{KZ}}](k_0) = \frac{n_0^2}{2\Omega^{(d)}} \int d\Omega_0 \int d^d k_1 \int d^d k_2 \; \tilde{w}(k_0 \to k_1 k_2)$$
$$\left[\left|\frac{k_0}{k_1}\right|^{s_0} \left|\frac{k_0}{k_2}\right|^{s_0} - \left(\left|\frac{k_0}{k_1}\right|^{s_0} + \left|\frac{k_0}{k_2}\right|^{s_0} \right) \right] \times \left[1 - \left|\frac{k_0}{k_1}\right|^{\xi} - \left|\frac{k_0}{k_2}\right|^{\xi} \right]$$

with $\xi = 2d - z + 2m - 2s_0$.

ii) Show that in the limit $s_0 \to m + d$ the energy flux $J_k(\Lambda, t) = -\int_0^\Lambda d^d k \; \omega(k) \; I[n](k,t)$ through a momentum shell is given by

$$J_k(\Lambda) = \frac{\omega_0 n_0^2 k_0^d}{2} \int d\Omega_0 \int d^d k_1 \int d^d k_2 \; \tilde{w}(k_0 \to k_1 k_2) \tag{49}$$
$$\times \left[\left|\frac{k_0}{k_1}\right|^{s_0} \left|\frac{k_0}{k_2}\right|^{s_0} - \left(\left|\frac{k_0}{k_1}\right|^{s_0} + \left|\frac{k_0}{k_2}\right|^{s_0} \right) \right] \left[\left|\frac{k_0}{k_1}\right|^{-z} \log \left|\frac{k_0}{k_1}\right| + \left|\frac{k_0}{k_2}\right|^{-z} \log \left|\frac{k_0}{k_2}\right| \right]$$

- From the in-class problem we showed that the energy fluxthrough a momentum shell is given by (with $\epsilon = 2m + 2d - 2s_0$)

$$J_k(\Lambda, t) = -\omega_0 k_0^{d-(2d-z+2m-2s_0+z)} \int d\Omega_0 \int_0^\Lambda dk \; k^{-2s_0+2d+2m-1} I[n_{\mathrm{KZ}}](k_0) \, , \tag{50}$$
$$= -\omega_0 k_0^{d+\epsilon} \frac{\Lambda^\epsilon}{\epsilon} \int d\Omega_0 I[n_{\mathrm{KZ}}](k_0) \, , \tag{51}$$
$$= -\omega_0 k_0^{d+\epsilon} \frac{\Lambda^\epsilon}{\epsilon} \frac{n_0^2}{2} \int d\Omega_0 \int d^d k_1 \int d^d k_2 \; \tilde{w}(k_0 \to k_1 k_2)$$
$$\left[\left|\frac{k_0}{k_1}\right|^{s_0} \left|\frac{k_0}{k_2}\right|^{s_0} - \left(\left|\frac{k_0}{k_1}\right|^{s_0} + \left|\frac{k_0}{k_2}\right|^{s_0} \right) \right] \times \left[1 - \left|\frac{k_0}{k_1}\right|^{\epsilon - z} - \left|\frac{k_0}{k_2}\right|^{\epsilon - z} \right]$$
$$= \frac{\omega_0 n_0^2 k_0^d}{2} \int d\Omega_0 \int d^d k_1 \int d^d k_2 \; \tilde{w}(k_0 \to k_1 k_2) \tag{52}$$
$$\times \left[\left|\frac{k_0}{k_1}\right|^{s_0} \left|\frac{k_0}{k_2}\right|^{s_0} - \left(\left|\frac{k_0}{k_1}\right|^{s_0} + \left|\frac{k_0}{k_2}\right|^{s_0} \right) \right] \left[\left|\frac{k_0}{k_1}\right|^{-z} \log \left|\frac{k_0}{k_1}\right| + \left|\frac{k_0}{k_2}\right|^{-z} \log \left|\frac{k_0}{k_2}\right| \right]$$

where we cancel $1 - \left|\frac{k_0}{k_1}\right|^{-z} - \left|\frac{k_0}{k_2}\right|^{-z}$ due to energy conservation.

Now consider as a particular example the dynamics of surface waves in shallow water, as described by a three wave kinetic equation in $d = 2$ dimensions. In this particular case, the dispersion relation is given by $\omega(k) = \omega_0 k^2 / k_0^2$ and the interaction matrix element takes a particularly simple form $V(k, k_1, k_2) = V_0 k^2 / k_0^2$, with $\omega_0 / k_0^2 = \left(\frac{\sigma h}{\rho}\right)^{1/2}$ and $V_0 / k_0^2 = \frac{1}{8\pi} \left(\frac{\sigma}{4\rho h}\right)^{1/4}$ where ρ and σ denote the density and surface tension of the fluid, and h is the height of the container.

iii) Determine the scaling exponent s_0 of the KZ spectrum for stationary turbulence of capillary waves on shallow water (Check: $s_0 = 4$)

- The different exponent are $m = 2$, $d = 2$ and $z = 2$.

- The scaling exponent are from the previous exercise is given by $s_0 = d + m = 4$.

iv) Show that for two-dimensional dynamics of capillary waves on shallow water, the phase-space integrations in Eq. (49) can be performed according to

$$\int d\Omega_0 \int d^2k_1 \int d^2k_2\ \tilde{w}(k_0 \to k_1 k_2) f\left(\left(\frac{k_1}{k_0}\right),\left(\frac{k_2}{k_0}\right)\right) = \frac{(2\pi)^2 k_0^2 V_0^2}{2\omega_0} \int_0^1 dx \frac{f\left(x, \sqrt{1-x^2}\right)}{\sqrt{1-x^2}} ,$$

- The energy flux integral is given by

$$J_k(\Lambda) = \frac{\omega_0 n_0^2 k_0^2}{2} \int d\Omega_0 \int d^2k_1 \int d^2k_2\ \tilde{w}(k_0 \to k_1 k_2) \tag{53}$$
$$\times \left[\left|\frac{k_0}{k_1}\right|^{s_0}\left|\frac{k_0}{k_2}\right|^{s_0} - \left(\left|\frac{k_0}{k_1}\right|^{s_0} + \left|\frac{k_0}{k_2}\right|^{s_0}\right)\right]\left[\left|\frac{k_0}{k_1}\right|^{-z}\log\left|\frac{k_0}{k_1}\right| + \left|\frac{k_0}{k_2}\right|^{-z}\log\left|\frac{k_0}{k_2}\right|\right] ,$$
$$J_k(\Lambda) = \frac{\omega_0 n_0^2 k_0^2}{2} \int d\Omega_0 \int d^2k_1 \int d^2k_2\ \tilde{w}(k_0 \to k_1 k_2) f\left(\left(\frac{k_1}{k_0}\right),\left(\frac{k_2}{k_0}\right)\right) , \tag{54}$$

where $f\left(\left(\frac{k_1}{k_0}\right),\left(\frac{k_2}{k_0}\right)\right) = \left[\left|\frac{k_0}{k_1}\right|^{s_0}\left|\frac{k_0}{k_2}\right|^{s_0} - \left(\left|\frac{k_0}{k_1}\right|^{s_0} + \left|\frac{k_0}{k_2}\right|^{s_0}\right)\right]\left[\left|\frac{k_0}{k_1}\right|^{-z}\log\left|\frac{k_0}{k_1}\right| + \left|\frac{k_0}{k_2}\right|^{-z}\log\left|\frac{k_0}{k_2}\right|\right]$ and $\widetilde{w}(k_0 \to k_1k_2) = (2\pi)\delta(\omega(k_0) - \omega(k_1) - \omega(k_2))\delta^{(2)}(k_0 - k_1 - k_2)V_0^2$.

- Using the momentum delta functions we can perform k_2 integral where

$$\vec{k}_2 = \vec{k}_0 - \vec{k}_1 = \begin{pmatrix} k_0\cos\theta_0 - k_1\cos\theta_1 \\ k_0\sin\theta_0 - k_1\sin\theta_1 \end{pmatrix} , \tag{55}$$
$$k_2^2 = k_0^2 + k_1^2 + 2k_0^2 x\cos(\theta_0 - \theta_1) , \tag{56}$$
$$\left|\frac{k_2}{k_0}\right|^2 = 1 + x^2 + 2x\cos(\theta_0 - \theta_1) . \tag{57}$$

We define $x = |\frac{k_1}{k_0}|$, then we have $|\frac{k_2}{k_0}| = \sqrt{1-x^2}$ and $d^2k_1 = k_1 dk_1 d\theta_1 = k_0^2 x dx d\theta_1$

$$f\left(\left(\frac{k_1}{k_0}\right),\left(\frac{k_2}{k_0}\right)\right) = f(x, \sqrt{1-x^2}) \tag{58}$$
$$\widetilde{w}(k_0 \to k_1k_2) = \frac{(2\pi)}{\omega_0}\delta\left(1 - x^2 - \left|\frac{k_2}{k_0}\right|^2\right)V_0^2 , \tag{59}$$
$$= \frac{(2\pi)}{\omega_0}\delta\underbrace{\left(-2x^2 - 2x\cos(\theta_0 - \theta_1)\right)}_{g(\theta_1)}V_0^2 . \tag{60}$$

Using θ_1 integral we can perform the delta function leading to thre replacement

$$\theta_1 \to \theta_0 - \arccos(-x) , \tag{61}$$

and we are left with

$$
\begin{aligned}
I &= \int d\Omega_0 \int d^2k_1 \int d^2k_2\ \tilde{w}(k_0 \to k_1 k_2) f(x, \sqrt{1-x^2})\,, && (62)\\
&= \frac{(2\pi)k_0^2 V_0^2}{\omega_0} \int_0^{2\pi} d\theta_0 d\theta_1 \int dx\ \delta(g(\theta_1)) x f(x, \sqrt{1-x^2})\,, && (63)\\
&= \frac{(2\pi)k_0^2 V_0^2}{\omega_0} \int_0^{2\pi} d\theta_0 \int dx\ x \frac{1}{|g'(\theta_1)|} f(x, \sqrt{1-x^2})\,, && (64)\\
&= \frac{(2\pi)^2 k_0^2 V_0^2}{\omega_0} \int dx\ x \frac{1}{2x \sin(\arccos(-x))} f(x, \sqrt{1-x^2})\,, && (65)\\
&= \frac{(2\pi)^2 k_0^2 V_0^2}{2\omega_0} \int_0^1 dx \frac{f\left(x, \sqrt{1-x^2}\right)}{\sqrt{1-x^2}}\,, && (66)
\end{aligned}
$$

v) Calculate the energy flux through a momentum shell $J_k(\Lambda)$ in Eq. (49) for stationary turbulence of capillary waves on shallow water (Check: $J_k(\Lambda) = 2\pi^3 n_0^2 k_0^4 V_0^2$)

- The energy flux is then given by

$$
\begin{aligned}
J_k(\Lambda) &= \frac{\omega_0 n_0^2 k_0^2}{2} \frac{(2\pi)^2 k_0^2 V_0^2}{2\omega_0} \int_0^1 dx \frac{f\left(x, \sqrt{1-x^2}\right)}{\sqrt{1-x^2}}\,,\\
&= 2\pi^3 n_0^2 k_0^4 V_0^2\,. && (67)
\end{aligned}
$$

vi) Show that if energy is injected into the system at a constant rate P, the stationary turbulent solution in the intertial range of the cascade takes the form

$$
n_{KZ}(k) = \frac{8P^{1/2}}{\sqrt{\pi}} \left(\frac{\rho h}{\sigma}\right)^{1/4} k^{-4}\,, \quad (68)
$$

- When enegy is injected at a constant rate P the energy flux is given by

$$
\begin{aligned}
P &= J_k(\Lambda) = 2\pi^3 n_0^2 k_0^4 V_0^2 = 2\pi^3 n_0^2 k_0^4 k_0^4 \frac{1}{64\pi^2} \left(\frac{\sigma}{4\rho h}\right)^{1/2}\,, && (69)\\
k_0^{-4} &= \sqrt{\frac{\pi}{64}} n_0 P^{-1/2} \left(\frac{\sigma}{\rho h}\right)^{1/4} && (70)
\end{aligned}
$$

- The stationary turbulent solution in the inertial range can be expressed as

$$
n_{\mathrm{KZ}}(k) = n_0 \left(\frac{k}{k_0}\right)^{-4} = \frac{8P^{1/2}}{\sqrt{\pi}} \left(\frac{\rho h}{\sigma}\right)^{1/4} k^{-4}\,. \quad (71)
$$

3.3 Formulae:

$$
x^{a+\epsilon} \simeq x^a + \epsilon x^a \log(x) + \mathcal{O}(\epsilon^2)\,,
$$

$$
\int_0^1 dx \frac{1}{\sqrt{1-x^2}} \left[x^{-4}\sqrt{(1-x^2)}^{-4} - \left(x^{-4} + \sqrt{1-x^2}^{-4}\right)\right] \left[x^2 \log\left(\frac{1}{x}\right) + (1-x^2)\log\left(\frac{1}{\sqrt{1-x^2}}\right)\right] = 2\pi
$$