Non-equilibrium physics WS 20/21 – Exercise Sheet 10:

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1 Discussion:

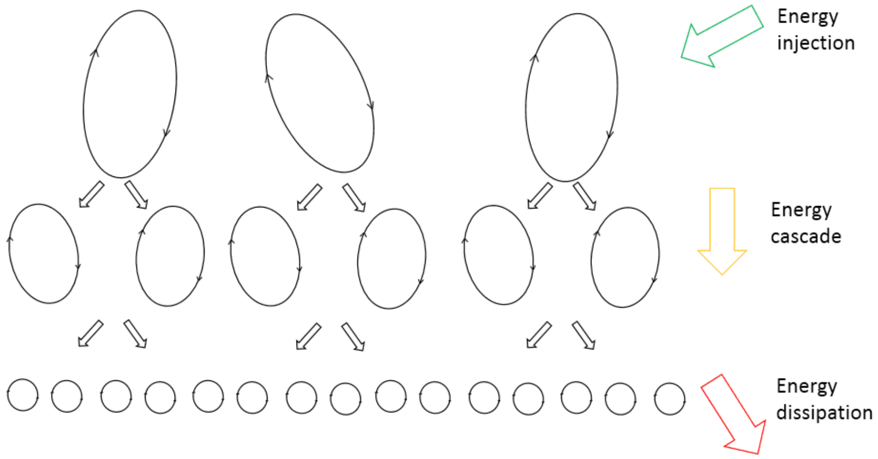

Fig. 1: Richardson cascade.

i) What is *stationary turbulence*? What is the *inertial range* of a turbulent cascade?

- Stationary : Describes a state where statistical properties of the system are time independent.
- Turbulence: Generally refers to phenomena associated with the transport of a conserved quantity across a large separation of scales.
- Stationary turbulence is then a state, where a conserved quantity is transported across a large separation of scales, in a way that despite this transport being dynamical, the statistical properties of the system are time independent. Stationary turbulent solutions associated with weak wave turbulence, correspond to time independent solutions of the wave-kinetic equation.
- The inertial range is the intermediate scale $l_{\text{source}} \gg l \gg l_{\text{sink}}$ away from the source and sink, where turbulence properties only depend on the rate of transfer of a conserved quantity. In scale invariant systems the stationary turbulent solutions in the inertial range of wavenumbers/momenta are described by the Kolmogorov-Zakharov spectra.

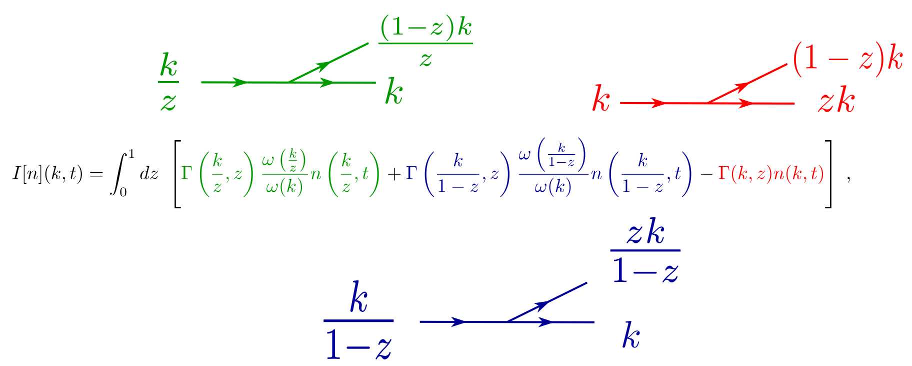

Fig. 2: Diagrammatic representation of the splitting collision kernel. (Note : $0 \leq z \leq 1$ and $0 \leq 1-z \leq 1 \Rightarrow \frac{k}{z} \geq k$ and $\frac{k}{1-z} \geq k$.)

2 In-class problems:

2.1 Collinear branching cascade

Consider a one-dimensional branching process, where particles with energy $\omega(k) = ck$ can split into fragments with energies $z\omega(k)$ and $(1-z)\omega(k)$ with a rate $\Gamma(k,z)$. Neglecting the inverse process, the dynamics of this branching process is described by a kinetic equation

$$\partial_t n(k,t) = I[n](k,t) \tag{1}$$

for the particle number density $n(k,t)$, with the collision kernel given by

$$I[n](k,t) = \int_0^1 dz \left[\Gamma\left(\frac{k}{z}, z\right) \frac{\omega\left(\frac{k}{z}\right)}{\omega(k)} n\left(\frac{k}{z}, t\right) + \Gamma\left(\frac{k}{1-z}, z\right) \frac{\omega\left(\frac{k}{1-z}\right)}{\omega(k)} n\left(\frac{k}{1-z}, t\right) - \Gamma(k,z)n(k,t) \right] , \tag{2}$$

i) Determine the spectral index s_0 of the stationary Kolmogorov solutions $n(k) = n_0 \left(\frac{k_0}{k}\right)^{s_0}$ for a scale invariant splitting rate of the form $\Gamma(k,z) = \gamma_0\ K(z) \left(\frac{k_0}{k}\right)^m$.

- We insert the stationary Kolmogorov solutions to the collision kernel

$$I[n](k,t) = \int_0^1 dz \left[\Gamma\left(\frac{k}{z}, z\right) \frac{\omega\left(\frac{k}{z}\right)}{\omega(k)} n\left(\frac{k}{z}, t\right) + \Gamma\left(\frac{k}{1-z}, z\right) \frac{\omega\left(\frac{k}{1-z}\right)}{\omega(k)} n\left(\frac{k}{1-z}, t\right) - \Gamma(k,z)n(k,t) \right] , \tag{3}$$

- The stationary solution is the zeros of the collision kernel determined by

$$0 = \gamma_0\ K(z) \left[\frac{1}{z} n_0 \left(\frac{zk_0}{k}\right)^{s_0+m} + \frac{1}{1-z} n_0 \left(\frac{(1-z)k_0}{k}\right)^{s_0+m} - n_0 \left(\frac{k_0}{k}\right)^{s_0+m} \right] , \tag{4}$$

$$0 = \gamma_0\ K(z) \left(\frac{k_0}{k}\right)^{s_0+m} n_0 \left[z^{s_0+m-1} + (1-z)^{s_0+m-1} - 1 \right] , \tag{5}$$

we find $s_0 = 2 - m$.

3 Homework problems:

3.1 Kolmogorov-Zakharov spectra for four wave interactions

Consider the four wave kinetic equation

$$\partial_t n(k,t) = I[n](k,t) \ ,$$

for a statistically homogenous and isotropic system of waves, with power law dispersion $\omega(k) = \omega_0(k/k_0)^z$ with the collision integral given by

$$I[n](k,t) = \frac{1}{2\Omega^{(d)}} \int d\Omega_k d\Omega_{k_1} d\Omega_{k_2} d\Omega_{k_3} \int dk_1 k_1^{d-1} \int dk_2 k_2^{d-1} \int dk_3 k_3^{d-1}$$
$$\times \widetilde{w}(kk_1 \to k_2k_3) n(k,t) n(k_1,t) n(k_2,t) n(k_3,t) \left[\frac{1}{n(k,t)} + \frac{1}{n(k_1,t)} - \frac{1}{n(k_2,t)} - \frac{1}{n(k_3,t)}\right]$$

where

$$\widetilde{w}(kk_1 \to k_2k_3) = (2\pi)\delta(\omega(k)+\omega(k_1)-\omega(k_2)-\omega(k_3))\delta^{(d)}(k+k_1-k_2-k_3)\frac{|T(k,k_1,k_2,k_3)|^2}{2} \tag{6}$$

for a scale invariant matrix element $T(\lambda k, \lambda k_1, \lambda k_2, \lambda k_3) = \lambda^m T(k,k_1,k_2,k_3)$ with the usual symmetry properties $T(k,k_1,k_2,k_3) = T(k_1,k,k_3,k_2) = T(k_2,k_3,k,k_1)$.

i) Show that for a scale invariant spectrum of the form $n(k) = n_0\left(\frac{k_0}{k}\right)^{s_0}$, the collision integral can be re-expressed in the following way by performing appropriate Zakharov transformations

$$I[n](k,t) = \frac{1}{2\Omega^{(d)}} \int d\Omega_k d\Omega_{k_1} d\Omega_{k_2} d\Omega_{k_3} \int dk_1 k_1^{d-1} \int dk_2 k_2^{d-1} \int dk_3 k_3^{d-1}$$
$$\times \widetilde{w}(kk_1 \to k_2k_3)\, n(k_1,t)\, n(k_2,t)\, n(k_3,t) \left[1 + \left(\frac{k}{k_1}\right)^x - \left(\frac{k}{k_2}\right)^x - \left(\frac{k}{k_3}\right)^x\right]$$

$\Big($Cross-check: $x = 2m - 3s_0 - d - z + 4d$ $\Big)$

- For a scale invariant spectrum the collision integral can be re-expressed as follows

$$I[n](k,t) = \frac{1}{2\Omega^{(d)}} \int d\Omega_k d\Omega_{k_1} d\Omega_{k_2} d\Omega_{k_3} \int dk_1 k_1^{d-1} \int dk_2 k_2^{d-1} \int dk_3 k_3^{d-1}$$
$$\times \widetilde{w}(kk_1 \to k_2k_3) n(k,t) n(k_1,t) n(k_2,t) n(k_3,t) \left[\frac{1}{n(k,t)} + \frac{1}{n(k_1,t)} - \frac{1}{n(k_2,t)} - \frac{1}{n(k_3,t)}\right] \ ,$$

Using the following Zakharov transformation

$$k = k_1' \frac{k}{k_1'} \ , \qquad k_1 = k\frac{k}{k_1'} \ , \qquad k_2 = k_2' \frac{k}{k_1'} \ , \qquad k_3 = k_3' \frac{k}{k_1'} \ , \tag{7}$$

we write the different terms of the integrand

$$
\begin{aligned}
dk_1 &= -\frac{k^2}{k_1'^2} dk_1' \,, \qquad dk_2 = \frac{k}{k_1'} dk_2' \,, \qquad dk_3 = \frac{k}{k_1'} dk_3' \,, && (8)\\
n(k,t) &= n(k_1',t) \left(\frac{k_1'}{k}\right)^{s_0} \qquad n(k_1,t) = n(k,t) \left(\frac{k_1'}{k}\right)^{s_0} && (9)\\
n(k_2,t) &= n(k_2',t) \left(\frac{k_1'}{k}\right)^{s_0} \qquad n(k_3,t) = n(k_3',t) \left(\frac{k_1'}{k}\right)^{s_0} && (10)\\
\delta(\omega(k)+\omega(k_1)-\omega(k_2)-\omega(k_3)) &= \delta\omega_0 \left(\left(\frac{k}{k_0}\right)^z + \left(\frac{k_1}{k_0}\right)^z - \left(\frac{k_2}{k_0}\right)^z - \left(\frac{k_3}{k_0}\right)^z \right) , && (11)\\
&= \delta\omega_0 \left(\left(\frac{k}{k_0}\right)^z + \left(\frac{k^2}{k_1' k_0}\right)^z - \left(\frac{k k_2'}{k_1' k_0}\right)^z - \left(\frac{k k_3'}{k_1' k_0}\right)^z \right) , && (12)\\
&= \left(\frac{k_1'}{k}\right)^z \delta(\omega(k_1')+\omega(k)-\omega(k_2')-\omega(k_3')) \,, && (13)\\
\delta^{(d)}(k+k_1-k_2-k_3) &= \left(\frac{k_1'}{k}\right)^d \delta(k_1'+k-k_2'-k_3') \,, && (14)\\
|T(k,k_1,k_2,k_3)|^2 &= \left(\frac{k_1'}{k}\right)^{-2m} |T(k_1',k,k_2',k_3')|^2 && (15)
\end{aligned}
$$

The integral becomes

$$
\begin{aligned}
&\frac{1}{2\Omega^{(d)}} \int d\Omega_k d\Omega_{k_1} d\Omega_{k_2} d\Omega_{k_3} \int dk_1 k_1^{d-1} \int dk_2 k_2^{d-1} \int dk_3 k_3^{d-1} \\
&\times \widetilde{w}(kk_1 \to k_2k_3) n(k,t) n(k_1,t) n(k_2,t) n(k_3,t) \frac{1}{n(k_1,t)} \,, && (16)\\
&= \frac{1}{2\Omega^{(d)}} \int d\Omega_k d\Omega_{k_1} d\Omega_{k_2} d\Omega_{k_3} \int dk_1' \left(-\frac{k^2}{k_1'^2}\right) \left(\frac{k^2}{k_1'}\right)^{d-1} \int dk_2' k_2'^{d-1} \left(\frac{k}{k_1'}\right)^d \int dk_3' k_3'^{d-1} \left(\frac{k}{k_1'}\right)^d \\
&\times \widetilde{w}(kk_1' \to k_2'k_3') n(k_1',t) n(k_2',t) n(k_3',t) \left(\frac{k_1'}{k}\right)^{3s_0} \left(\frac{k_1'}{k}\right)^{z+d-2m} \,, && (17)\\
&= \frac{1}{2\Omega^{(d)}} \int d\Omega_k d\Omega_{k_1} d\Omega_{k_2} d\Omega_{k_3} \int dk_1' k_1'^{d-1} \int dk_2' k_2'^{d-1} \int dk_3' k_3'^{d-1} \\
&\times \widetilde{w}(kk_1' \to k_2'k_3') n(k_1',t) n(k_2',t) n(k_3',t) \left(\frac{k}{k_1'}\right)^{2m-3s_0-z-d+4d} \,. && (18)\\
& && (19)
\end{aligned}
$$

Analogously for the other momenta one finds

$$
\begin{aligned}
I[n](k,t) = \frac{1}{2\Omega^{(d)}} &\int d\Omega_k d\Omega_{k_1} d\Omega_{k_2} d\Omega_{k_3} \int dk_1 k_1^{d-1} \int dk_2 k_2^{d-1} \int dk_3 k_3^{d-1} \\
&\times \widetilde{w}(kk_1 \to k_2k_3)\, n(k_1,t)\, n(k_2,t)\, n(k_3,t) \left[1 + \left(\frac{k}{k_1}\right)^x - \left(\frac{k}{k_2}\right)^x - \left(\frac{k}{k_3}\right)^x \right] . && (20)
\end{aligned}
$$

with $x = 2m - 3s_0 - z - d + 4d$.

ii) Determine the scaling exponent s_0 for the stationary Kolmogorov solutions of the four wave kinetic equation. How many solutions can you find? Which conserved quantity is transported in the turbulent cascade for the different cases?

- The scaling exponent s_0 solves the following equation

$$
k^{-x} + k_1^{-x} = k_2^{-x} + k_3^{-x} \,, \qquad (21)
$$

which cancels due to particle number and energy conservation

$$\text{Particle number } x = 0 \qquad s_0 = (2m - z + 3d)/3 \,, \tag{22}$$
$$\text{Energy } x = -z \qquad s_0 = (2m + 3d)/3 \,. \tag{23}$$

3.2 Formulae:

$$x^{a+\epsilon} \simeq x^a + \epsilon x^a \log(x) + \mathcal{O}(\epsilon^2) \,,$$

$$\int_0^1 dz[(1-z)\log(1-z) + z\log(z)] = -\frac{1}{2} \,,$$
$$\int_0^1 dz[(1-z)\log(1-z) + z\log(z)]\frac{8}{\pi}\sqrt{z(1-z)} = \frac{5}{6} - 2\log(2) \approx -0.552961$$
$$\int_0^1 dz[(1-z)\log(1-z) + z\log(z)]\frac{1}{\pi\sqrt{z(1-z)}} = 1 - \log(4) \approx -0.386294$$

3.3 Collinear branching cascade (continued)

Consider again the one-dimensional branching process in Eqns. (1,2).

i) Show that for $n(k) = n_0 \left(\frac{k_0}{k}\right)^{s_0}$ with $s_0 > 2 - m$ the energy flux $\dot{E}(\Lambda) = \int_\Lambda^\infty dk\ \omega(k)\ I[n](k,t)$ can be expressed as

$$\dot{E}(\Lambda) = \gamma_0\ n_0 k_0\ c k_0 \left(\frac{k_0}{\Lambda}\right)^{m+s_0-2} \int_0^1 dz\ \frac{z^{m+s_0-1} + (1-z)^{m+s_0-1} - 1}{m+s_0-2}\ K(z) \,,$$

- The energy flux is written

$$\dot{E}(\Lambda) = \int_\Lambda^\infty dk\ \omega(k) \int_0^1 dz \left[\Gamma\left(\frac{k}{z}, z\right)\frac{\omega\left(\frac{k}{z}\right)}{\omega(k)} n\left(\frac{k}{z}, t\right)\right. \tag{24}$$
$$\left. + \Gamma\left(\frac{k}{1-z}, z\right)\frac{\omega\left(\frac{k}{1-z}\right)}{\omega(k)} n\left(\frac{k}{1-z}, t\right) - \Gamma(k,z) n(k,t)\right] \,, \tag{25}$$
$$= \int_\Lambda^\infty dk \int_0^1 dz\ \gamma_0\ K(z)\left(\frac{k_0}{k}\right)^{s_0+m} k n_0 x \left[z^{s_0+m-1} + (1-z)^{s_0+m-1} - \right] \,, \tag{26}$$
$$= \gamma_0\ n_0 k_0\ c k_0 \left(\frac{k_0}{\Lambda}\right)^{m+s_0-2} \int_0^1 dz\ \frac{z^{m+s_0-1} + (1-z)^{m+s_0-1} - 1}{m+s_0-2}\ K(z) \,. \tag{27}$$

ii) Evaluate the dimensionless integral over z using L'Hospital's rule to take the limit $\epsilon = m + s_0 - 2$ going to zero, where the spectrum $n(k)$ approaches the stationary Kolmogorov solution. Show that in this limit the stationary flux becomes scale (Λ) independent and is expliclity given by

$$\dot{E}(\Lambda) = \gamma_0\ n_0 k_0\ c k_0 \int_0^1 dz\ \left[z\log(z) + (1-z)\log(1-z)\right] K(z) \tag{28}$$

What can you say about the sign of $\dot{E}(\Lambda)$ (assuming that the splitting kernel $K(z)$ is positive semi-definite), and what does this tell you about the direction of the cascade?

- Using L'Hospital's rule to evaluate the limit $\epsilon = m + s_0 - 2 \to 0$ the energy flux becomes

$$\dot{E}(\Lambda) = \lim_{\epsilon\to 0} \gamma_0\; n_0 k_0\; ck_0 \left(\frac{k_0}{\Lambda}\right)^{m+s_0-2} \int_0^1 dz\; \frac{z^{\epsilon+1} + (1-z)^{\epsilon+1} - 1}{\epsilon}\, K(z)\;, \tag{29}$$

$$= \gamma_0\; n_0 k_0\; ck_0 \int_0^1 dz\; \frac{\lim_{\epsilon\to 0} \partial_\epsilon \left[z^{\epsilon+1} + (1-z)^{\epsilon+1} - 1\right]}{\lim_{\epsilon\to 0} \partial_\epsilon \epsilon}\, K(z)\;, \tag{30}$$

$$= \gamma_0\; n_0 k_0\; ck_0 \int_0^1 dz\; \left[z\log(z) + (1-z)\log(1-z)\right] K(z)\;. \tag{31}$$

For $0 \le z \le 1$ we have $z\log z \le 0$, leading to a negative energy flux, i.e. an inverse energy cascade from high energies to low energies.

iii) Evaluate the stationary energy flux $\dot{E}(\Lambda)$ in Eq. (28) for the splitting kernels

$$K(z) = 1\;, \qquad K(z) = \frac{1}{\pi\sqrt{z(1-z)}}\;, \qquad \text{and} \qquad K(z) = \frac{8}{\pi}\sqrt{z(1-z)} \tag{32}$$

Which kernel is most efficient in transporting energy? Why? $\Big($Note that all of the above kernels are normalized such that the integrated splitting rate $\int_0^1 dz\; \Gamma(k,z) = \gamma_0\; \left(\frac{k_0}{k}\right)^m$ is identical. $\Big)$

- The energy flux for each kernel is

$$\dot{E}(\Lambda) = \gamma_0\; n_0 k_0\; ck_0 \int_0^1 dz\; \left[z\log(z) + (1-z)\log(1-z)\right] K(z)\;, \tag{33}$$

$$= \gamma_0\; n_0 k_0\; ck_0 \begin{cases} -0.5 & \text{for } K(z) = 1 \\ -0.386294 & \text{for } K(z) = \frac{1}{\pi\sqrt{z(1-z)}} \\ -0.552961 & \text{for } K(z) = \frac{8}{\pi}\sqrt{z(1-z)} \end{cases} \tag{34}$$

The kernel $K(z) = \frac{8}{\pi}\sqrt{z(1-z)}$ is the most efficient because it favors democratic splitting with energy fraction $z = (1-z) = 1/2$ which is the most efficient way to transport energy.

We will now focus on the case of a constant splitting kernel $K(z) = 1$ and consider the specific situation where energy is injected into the system with a constant rate $\dot{E}_{\rm in}$ at a characteristic momentum scale $k_{\rm in}$, and removed from the system with the same rate $\dot{E}_{\rm out} = \dot{E}_{\rm in}$ at a much smaller momentum scale $k_{\rm out} \ll k_{\rm in}$.

iv) Determine the exact form of the stationary Kolmogorov solution $n(k)$ (including the non-universal amplitude n_0) in the inertial range of momenta $k_{\rm out} \ll k \ll k_{\rm in}$.

- We have $s_0 = 2 - m$. The stationary Kolmogorov solution $n(k)$ is

$$n(k) = n_0 \left(\frac{k_0}{k}\right)^{2-m}\;. \tag{35}$$

Since the conserved quantity transported in the turbulent cascade is energy, the energy flux through the cascade is scale independent and equal to $\dot{E}_{\rm in}$

$$\dot{E}(\Lambda) = \gamma_0\; n_0 k_0\; ck_0 \int_0^1 dz\; \left[z\log(z) + (1-z)\log(1-z)\right] K(z)\;, \tag{36}$$

$$= -\frac{1}{2}\gamma_0\; n_0 k_0\; ck_0 = \dot{E}_{\rm in}\;. \tag{37}$$

Using the result for n_0 we write

$$n(k) = -\frac{2\dot{E}_{\rm in}}{\gamma_0 c k^{2-m}}\, (k_0)^{-m}\;. \tag{38}$$