

# Non-equilibrium physics WS 20/21 – Exercise Sheet 1:

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## 1 Discussion:

i) Briefly discuss the meaning of the *local equilibrium assumption* for discrete and continuous systems

- For a discrete system, we can divide the full system into subsystems/cells, considered in equilibrium. Each cell can be of arbitrary size. The physics inside the cell  $i$  is then described by macroscopic extensive quantities relevant to thermodynamic equilibrium ( $\{X^{(a)}\} \equiv U^{(a)}, V^{(a)}, N^{(a)}$ ). In addition, the equilibrium definition of entropy holds inside each cell

$$dS^{(a)} = \sum_i \underbrace{\frac{\partial S^{(a)}}{\partial X_i^{(a)}}}_{Y_i^{(a)}} dX_i^{(a)} = \frac{1}{T^{(a)}} dU^{(a)} + \frac{p^{(a)}}{T^{(a)}} dV_i - \frac{\mu^{(a)}}{T^{(a)}} dN^{(a)}. \quad (1)$$

- The same holds, in the continuous case, but now we take the limit of infinitely small cells and define densities of the macroscopic quantities described by functions of space ( and time) i.e. ( $\{x_i(\vec{r}, t)\} \equiv T(\vec{r}, t), p(\vec{r}, t), \mu(\vec{r}, t)$ ). The entropy density is now written as a function of space and time as well

$$ds(\vec{r}, t) = \sum_i \underbrace{\frac{\partial s(\vec{r}, t)}{\partial x_i}}_{y_i(\vec{r}, t)} dx_i(\vec{r}, t) = \frac{1}{T(\vec{r}, t)} d\epsilon(\vec{r}, t) - \frac{\mu(\vec{r}, t)}{T(\vec{r}, t)} dn(\vec{r}, t). \quad (2)$$

Here we talk about a system with different timescales, a fast timescale (e.g. elastic collisions) driving particles to local equilibrium and a slow timescale (e.g. external force) driving the whole system to global equilibrium. We will discuss an example in future lecture in the context of a plasma following a Boltzmann kinetic equation.

ii) Briefly discuss the meaning of *affinities and fluxes* for equilibrium and non-equilibrium systems

- When discrete cells in local equilibrium are allowed to exchange quantities (e.g. T, p, N..) this leads to a flux of the quantity from one cell to the other. As entropy is additive, the total entropy is the sum of the entropy of each cell. Due to conservation laws of the total system of all cells, the total entropy only depends on extensive quantities of one system  $\{X_j^{(a)}\}$ . One finds that the entropy is written (as we will show in following exercise)

$$\frac{dS^{\text{tot}}}{dt} = \sum_i \underbrace{\frac{dS^{\text{tot}}}{dX_i^{(a)}}}_{F_i} \underbrace{\frac{dX_j^{(a)}}{dt}}_{J_j} = \sum_j F_i J_i, \quad (3)$$

where  $F_j$  are the affinities or general forces driving the system to global equilibrium and  $J_j$  are fluxes of macroscopic quantity  $X_j$  describing the flow of these quantities as a consequence of the affinity.

- Analogously to the discrete system, in a continuous system the conservation laws lead to the following expression for the entropy

$$\frac{\partial s(\vec{r}, t)}{\partial t} + \vec{\nabla} \cdot \vec{\mathcal{J}}_s(\vec{r}, t) = \sigma_S(\vec{r}, t), \quad (4)$$

where  $\vec{\mathcal{J}}_s(\vec{r}, t)$  is the entropy flux density and  $\sigma_S(\vec{r}, t)$  is the entropy production rate

$$\sigma_S(\vec{r}, t) = \sum_j \vec{F}_j(\vec{r}, t) \cdot \vec{\mathcal{J}}_j(\vec{r}, t). \quad (5)$$

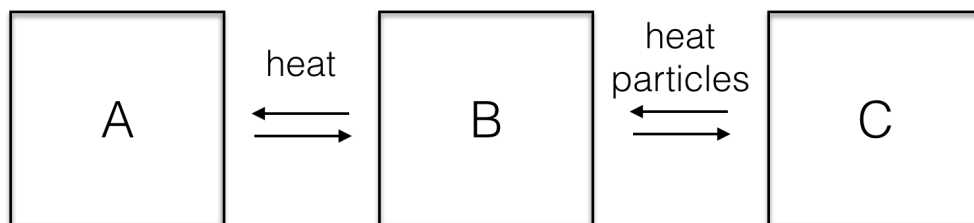
Now, both the affinities and fluxes are continuous, we write

$$\vec{F}_j(\vec{r}, t) = \vec{\nabla} y_i(\vec{r}, t). \quad (6)$$

## 2 In-class problems:

### 2.1 Containers

Consider a system composed of three isolated sub-systems  $A, B, C$ . Initially each system is in equilibrium, with internal energies  $U_0^A, U_0^B, U_0^C$ , containing  $N_0^A, N_0^B, N_0^C$  molecules of an ideal gas in volumes  $V_0^A, V_0^B, V_0^C$ . Subsequently, the isolation is removed in such a way that sub-systems  $A$  and  $B$  can only exchange heat with each other, while sub-systems  $B$  and  $C$  can exchange both particle number and heat.



- Express the global conservation laws and remaining constraints and deduce the possible variations of the extensive parameters

- Number conservation in A :  $\frac{dN_A}{dt} = 0 \Rightarrow N_A(t) = N_0^A$
- Number conservation between B and C :  $\frac{dN_B}{dt} + \frac{dN_C}{dt} = 0$
- Total energy conservation :  $\frac{dU_A}{dt} + \frac{dU_B}{dt} + \frac{dU_C}{dt} = 0$
- Volume constraints :  $\frac{dV_A}{dt} = \frac{dV_B}{dt} = \frac{dV_C}{dt} = 0$

- Based on the possible variations of the extensive parameters, construct the expression for the entropy production rate  $dS^{\text{tot}}/dt$

- Based on *Gibbs' fundamental equation* :  $dS = \frac{1}{T}dU + \frac{p}{T}dV - \frac{\mu}{T}dN$ , we write the entropy production rate:

$$\frac{dS^{\text{tot}}}{dt} = \frac{dS_A}{dt} + \frac{dS_B}{dt} + \frac{dS_C}{dt} \quad (7)$$

$$= \frac{1}{T_A} \frac{dU_A}{dt} + \frac{1}{T_B} \frac{dU_B}{dt} + \frac{1}{T_C} \frac{dU_C}{dt} - \frac{\mu_B}{T_B} \frac{dN_B}{dt} - \frac{\mu_C}{T_C} \frac{dN_C}{dt} \quad (8)$$

$$(9)$$

Using  $\frac{dU_C}{dt} = -\frac{dU_A}{dt} - \frac{dU_B}{dt}$  and  $\frac{dN_C}{dt} = -\frac{dN_B}{dt}$  from last exercise, we write

$$\frac{dS^{\text{tot}}}{dt} = \underbrace{\left(\frac{1}{T_A} - \frac{1}{T_C}\right)}_{F_U^{A \leftrightarrow C}} \frac{dU_A}{dt} + \underbrace{\left(\frac{1}{T_B} - \frac{1}{T_C}\right)}_{F_U^{B \leftrightarrow C}} \frac{dU_B}{dt} - \underbrace{\left(\frac{\mu_B}{T_B} - \frac{\mu_C}{T_C}\right)}_{F_N^{B \leftrightarrow C}} \frac{dN_B}{dt} \quad (10)$$

- iii) Determine the equilibrium conditions for the system in terms of the intensive quantities  $T^{(i)}, p^{(i)}, \mu^{(i)}$  in each sub-system

- In equilibrium we have  $\frac{dS^{\text{tot}}}{dt} = 0$ , Considering all allowed variations  $\frac{dX^{(a)}}{dt}$  the following terms vanish

$$\left(\frac{1}{T_A} - \frac{1}{T_C}\right) = 0 \Rightarrow T_A = T_C, \quad (11)$$

$$\left(\frac{1}{T_B} - \frac{1}{T_C}\right) = 0 \Rightarrow T_A = T_B = T_C, \quad (12)$$

$$\left(\frac{\mu_B}{T_B} - \frac{\mu_C}{T_C}\right) = 0 \Rightarrow \mu_B = \mu_C. \quad (13)$$

Lastly, the volume of each cell does not change.

## 2.2 Ideal gas

Consider a a classical ideal gas of  $N$  identical particles described by the non-interacting Hamiltonian  $H_N(\{x\}, \{p\}) = \sum_{i=1}^N \frac{p_i^2}{2m}$ .

- i) Calculate the canonical partition function

$$Z_C(T, N, V) = \frac{1}{N!} \int \left( \prod_{i=1}^N \frac{d^3 \vec{x}_i d^3 \vec{p}_i}{(2\pi\hbar)^3} \right) e^{-\frac{H_N(\{x\}, \{p\})}{k_B T}}$$

and determine the internal energy  $U = k_B T^2 \frac{\partial}{\partial T} \log Z_C(T, N, V)$ .

(Cross-check:  $U = \frac{3}{2} N k_B T$ )

- Due to the lack of interactions, the canonical partition function can be factorized as follows

$$Z_C(T, N, V) = \frac{1}{N!} \left( \prod_{i=1}^N \int \frac{d^3 \vec{x}_i d^3 \vec{p}_i}{(2\pi\hbar)^3} e^{-\frac{h_i(\{x\}, \{p\})}{k_B T}} \right) = \frac{1}{N!} (z)^N, \quad (14)$$

where  $h_i(\{x\}, \{p\})$  is the single particle Hamiltonian written

$$h_i(\{x\}, \{p\}) = \frac{\vec{p}^2}{2m}. \quad (15)$$

And  $z$  is the integral

$$z = \int \frac{d^3\vec{x} d^3\vec{p}}{(2\pi\hbar)^3} e^{-\frac{1}{k_B T} \frac{\vec{p}^2}{2m}} = \frac{V 4\pi}{(2\pi\hbar)^3} \int_0^\infty dp p^2 e^{-\frac{1}{k_B T} \frac{p^2}{2m}}, \quad (16)$$

$$\stackrel{\text{footnote 1}}{=} V \frac{(2mk_B T)^{3/2} \sqrt{\pi}}{2\pi^2 \hbar^3} \frac{1}{4}, \quad (17)$$

$$= V \left( \frac{mk_B T}{2\pi\hbar^2} \right)^{3/2}, \quad (18)$$

The full canonical partition function becomes

$$Z_C(T, N, V) = \frac{V^N}{N!} \left( \frac{mk_B T}{2\pi\hbar^2} \right)^{3N/2}. \quad (19)$$

The internal energy is then

$$U = k_B T^2 \frac{\partial}{\partial T} \log Z_C(T, N, V) \stackrel{\text{footnote 2}}{=} k_B T^2 \frac{3N}{2} \frac{\partial}{\partial T} \log T = \frac{3}{2} N k_B T. \quad (20)$$

ii) Calculate the grand-canonical partition function based on the expansion

$$Z_{GC}(T, \mu, V) = \sum_{N=0}^{\infty} e^{\frac{\mu}{k_B T} N} Z_C(T, N, V),$$

and determine the average particle number  $N = k_B T \frac{\partial}{\partial \mu} \log Z_{GC}(T, \mu, V)$ .

$$\left( \text{Cross-check: } N = V \left( \frac{mk_B T}{2\pi\hbar^2} \right)^{3/2} e^{\frac{\mu}{k_B T}} \right)$$

- Following the same approach as the previous example, the canonical partition function can be factorized as follows

$$Z_{GC}(T, N, V) = \frac{1}{N!} \left( \prod_{i=1}^N e^{\frac{\mu}{k_B T}} \int \frac{d^3\vec{x}_i d^3\vec{p}_i}{(2\pi\hbar)^3} e^{-\frac{h_i(\{x\}, \{p\})}{k_B T}} \right) = \sum_{N=0}^{\infty} \frac{1}{N!} (z e^{\frac{\mu}{k_B T}})^N, \quad (21)$$

$$= \exp \left\{ z e^{\frac{\mu}{k_B T}} \right\} \quad (22)$$

where  $z$  is the integral

$$z = V \left( \frac{mk_B T}{2\pi\hbar^2} \right)^{3/2}, \quad (23)$$

The grand canonical partition function becomes

$$Z_{GC}(T, N, V) = \exp \left\{ V \left( \frac{mk_B T}{2\pi\hbar^2} \right)^{3/2} e^{\frac{\mu}{k_B T}} \right\}. \quad (24)$$

The average number is written

$$N = k_B T \frac{\partial}{\partial \mu} \log Z_{GC}(T, \mu, V). \quad (25)$$

$$= k_B T \frac{\partial}{\partial \mu} \left\{ V \left( \frac{mk_B T}{2\pi\hbar^2} \right)^{3/2} e^{\frac{\mu}{k_B T}} \right\} = V \left( \frac{mk_B T}{2\pi\hbar^2} \right)^{3/2} e^{\frac{\mu}{k_B T}}. \quad (26)$$

- iii) Based on your results, determine the equations of state  $T = T(U, V, N)$  and  $\mu = \mu(U, V, N)$  expressing the intensive quantities  $T, \mu$  in terms of the extensive variables  $U, V, N$

$$\left( \text{Cross-check: } k_B T = \frac{2}{3} U/N \text{ and } \frac{\mu}{k_B T} = -\log \left[ \frac{V}{N} \left( \frac{mU/N}{3\pi\hbar^2} \right)^{3/2} \right] \right)$$

- Using the identities for internal energy and average number

$$U = \frac{3}{2} N k_B T, \quad (27)$$

$$N = V \left( \frac{m k_B T}{2\pi\hbar^2} \right)^{3/2} e^{\frac{\mu}{k_B T}}, \quad (28)$$

we write the EOS

$$T(U, V, N) = \frac{2}{3} \frac{U}{N k_B}, \quad (29)$$

$$\mu(U, V, N) = k_B T \log \left\{ \frac{N}{V} \left( \frac{mU/N}{3\pi\hbar^2} \right)^{-3/2} \right\}. \quad (30)$$

- iv) Exploit the relation between the grand-canonical partition function  $-k_B T \log Z_{GC} = \phi_G$  and the grand-potential  $\phi_G = U - TS - \mu N$  to derive the Sackur-Tetrode equation for the entropy of an ideal gas

$$S(U, V, N) = N k_B \left\{ \frac{5}{2} + \log \left[ \frac{V}{N} \left( \frac{4\pi m U}{3N(2\pi\hbar)^2} \right)^{3/2} \right] \right\}$$

- Equating the two definitions of the grand-potential we find the entropy of an ideal gas

$$S(U, V, N) = k_B \log Z_{GC} + \frac{U}{T} - \frac{\mu}{T} N, \quad (31)$$

$$= k_B \left\{ V \left( \frac{m k_B T}{2\pi\hbar^2} \right)^{3/2} e^{\frac{\mu}{k_B T}} \right\} + \frac{3}{2} N k_B - N k_B \log \left\{ \frac{N}{V} \left( \frac{mU/N}{3\pi\hbar^2} \right)^{-3/2} \right\}, \quad (32)$$

replacing  $\mu$  in the exponential with Eq. (30) cancels<sup>3</sup>, and we are left with

$$S(U, V, N) = N k_B + \frac{3}{2} N k_B - N k_B \log \left\{ \frac{N}{V} \left( \frac{mU/N}{3\pi\hbar^2} \right)^{-3/2} \right\}, \quad (33)$$

$$= N k_B \left\{ \frac{5}{2} + \log \left[ \frac{V}{N} \left( \frac{4\pi m U}{3N(2\pi\hbar)^2} \right)^{3/2} \right] \right\}, \quad (34)$$

- v) Based on the entropy function calculate the equation of state  $p = p(U, V, N)$  for the pressure of an ideal gas. Convince yourself as a cross check that you re-produce the result  $pV = N k_B T$ .

- The pressure is given by

$$p = T \frac{\partial S}{\partial V} = N k_B T \frac{\partial}{\partial V} \log(V), \quad (35)$$

$$p = \frac{N k_B T}{V}. \quad (36)$$

<sup>1</sup> We make use of the integral  $\int dx x^2 e^{-ax^2} = \frac{\sqrt{\pi}}{4a^{3/2}}$ .

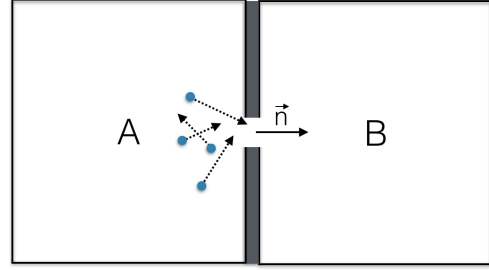
<sup>2</sup> We make use of the simple fact that  $\frac{\partial \log(x^\alpha y)}{\partial x} = \alpha \frac{\partial \log(x)}{\partial x} + \underbrace{\frac{\partial \log(y)}{\partial x}}_{=0} = \frac{\alpha}{x}$

<sup>3</sup> We have  $V \left( \frac{m k_B T}{2\pi\hbar^2} \right)^{3/2} e^{\frac{\mu}{k_B T}} = V \left( \frac{m k_B T}{2\pi\hbar^2} \right)^{3/2} \frac{N}{V} \left( \frac{mU/N}{3\pi\hbar^2} \right)^{-3/2} = N$ .

### 3 Homework problems:

#### 3.1 Heat and particle transfer between containers

Consider two containers  $A$  and  $B$  of identical volumes  $V_A = V_B$  each filled with a classical ideal gas. Initially both systems are in equilibrium at temperatures  $T_A$  and  $T_B$  and chemical potentials  $\mu_A$  and  $\mu_B$ . Now a small hole of cross-section  $S$  is drilled in the wall separating the containers  $A$  and  $B$ , allowing for exchange of particle number and heat between the two containers. If we assume that the particle density is uniform in each container, the amount of particles flowing through the hole can be calculated as



$$J_N^{A \rightarrow B} = \int \frac{d^3 \vec{p}}{(2\pi\hbar)^3} \int_S d^2 \vec{x} \frac{\vec{n} \cdot \vec{p}}{m} f(\vec{p}) \theta(\vec{n} \cdot \vec{p}),$$

where  $\int_S d^2 \vec{x}$  denotes the surface integral over the area of the hole,  $\vec{n}$  is the direction normal to the surface and  $f(\vec{p}) = e^{\frac{\mu}{k_B T}} e^{-\frac{\vec{p}^2}{2mk_B T}}$  is the Maxwell-Boltzmann distribution for the momenta of particles in an ideal gas.

i) Show that the particle flux  $J_N^{A \rightarrow B}$  through the hole from container  $A$  to container  $B$  is given by

$$J_N^{A \rightarrow B} = \frac{1}{\sqrt{2\pi m k_B T_A}} S p_A$$

- The particle flux through the hole from container  $A$  to  $B$  is written

$$J_N^{A \rightarrow B} = \int \frac{d^3 \vec{p}}{(2\pi\hbar)^3} \int_S d^2 \vec{x} \frac{\vec{n} \cdot \vec{p}}{m} f(\vec{p}) \theta(\vec{n} \cdot \vec{p}), \quad (37)$$

$$= S e^{\frac{\mu_A}{k_B T_A}} \frac{1}{(2\pi\hbar)^3} \int_0^\infty p^2 dp \int_0^1 d \cos \theta \int_0^{2\pi} d\phi \left( \frac{p \cos \theta}{m} \right) e^{-\frac{p^2}{2mk_B T_A}}. \quad (38)$$

Using Eq. (30) and performing the integration, we obtain

$$J_N^{A \rightarrow B} = S \left\{ \frac{N}{V} \left( \frac{m U_A / N_A}{3\pi\hbar^2} \right)^{-3/2} \right\} \frac{1}{m(2\pi\hbar)^3} 2\pi (m k_B T_A)^2, \quad (39)$$

$$= S \left\{ \frac{N}{V} \left( \frac{m k_B T_A}{2\pi\hbar^2} \right)^{-3/2} \right\} \frac{2\pi}{(2\pi\hbar)^3} m (k_B T_A)^2, \quad (40)$$

$$= \frac{1}{\sqrt{2\pi m k_B T_A}} S p_A, \quad (41)$$

ii) Determine the analogous integral expression for the energy flux  $J_E^{A \rightarrow B}$  through the hole from container  $A$  to container  $B$ . Show that  $J_E^{A \rightarrow B}$  is given by

$$J_E^{A \rightarrow B} = \sqrt{\frac{2k_B T_A}{\pi m}} S p_A$$

- The energy flux through the hole from container A to B is written

$$J_E^{A \rightarrow B} = \int \frac{d^3 \vec{p}}{(2\pi\hbar)^3} \int_S d^2 \vec{x} \frac{p^2}{2m} \left( \frac{\vec{n} \cdot \vec{p}}{m} \right) f(\vec{p}) \theta(\vec{n} \cdot \vec{p}), \quad (42)$$

$$= S \left\{ \frac{N}{V} \left( \frac{mk_B T_A}{2\pi\hbar^2} \right)^{-3/2} \right\} \frac{1}{(2\pi\hbar)^3} \int_0^\infty p^2 dp \int_0^1 d \cos \theta \int_0^{2\pi} d\phi \frac{p^2}{2m} \left( \frac{p \cos \theta}{m} \right) e^{-\frac{p^2}{2mk_B T}}, \quad (43)$$

$$= \sqrt{\frac{2k_B T_A}{\pi m}} S p_A \quad (44)$$

- iii) Determine the explicit form of the differential equations

$$\frac{d}{dt} N^A = J_N^{B \rightarrow A} - J_N^{A \rightarrow B}, \quad \frac{d}{dt} U^A = J_E^{B \rightarrow A} - J_E^{A \rightarrow B},$$

which govern the relaxation of the particle number and the internal energy, assuming that both sub-systems remain in equilibrium throughout the process.

- We write the change in the average number and internal energy of the container A as

$$\frac{d}{dt} N^A = J_N^{B \rightarrow A} - J_N^{A \rightarrow B}, \quad (45)$$

$$= \frac{1}{\sqrt{2\pi mk_B T_B}} S p_B - \frac{1}{\sqrt{2\pi mk_B T_A}} S p_A, \quad (46)$$

$$= \frac{S}{\sqrt{2\pi mk_B}} \left( \frac{p_B}{\sqrt{T_B}} - \frac{p_A}{\sqrt{T_A}} \right), \quad (47)$$

$$\frac{d}{dt} U^A = J_E^{B \rightarrow A} - J_E^{A \rightarrow B}, \quad (48)$$

$$= \sqrt{\frac{2k_B T_B}{\pi m}} S p_B - \sqrt{\frac{2k_B T_A}{\pi m}} S p_A, \quad (49)$$

$$= S \sqrt{\frac{2k_B}{\pi m}} \left( p_B \sqrt{T_B} - p_A \sqrt{T_A} \right), \quad (50)$$

- iv) Expressing the internal energy and particle number in each sub-system in terms of the globally conserved quantity  $N^{\text{tot}} = N^A + N^B$  and the difference between  $\Delta N = N^A - N^B$  as  $N^{A/B} = \frac{1}{2}(N^{\text{tot}} \pm \Delta N)$  (and similarly for  $U^{A/B}$ ), determine the equations of motion governing the relaxation of  $\Delta N$  and  $\Delta U$ . Linearize the equations of motion for  $\Delta N \ll N^{\text{tot}}$  and  $\Delta U \ll U^{\text{tot}}$  and determine the relaxation rates.

- $\Delta N$  obeys the following differential equation

$$\frac{d}{dt} \Delta N = \frac{d}{dt} N^A - \frac{d}{dt} N^B, \quad (51)$$

$$= 2 (J_N^{B \rightarrow A} - J_N^{A \rightarrow B}), \quad (52)$$

$$= 2 \frac{S}{V \sqrt{2\pi mk_B}} \left( \frac{N_B k_B T_B}{\sqrt{T_B}} - \frac{N_A k_B T_A}{\sqrt{T_A}} \right), \quad (53)$$

$$(54)$$

Using  $N^{A/B} = \frac{1}{2} (N^{\text{tot}} \pm \Delta N)$  and  $T_{A/B} = \frac{2U_{A/B}}{3k_B N_{A/B}} = \frac{2(U^{\text{tot}} \pm \Delta U)}{3k_B (N^{\text{tot}} \pm \Delta N)}$

$$\frac{d}{dt} \Delta N = 2 \frac{S}{V \sqrt{2\pi m}} \left( \frac{1}{2} (N^{\text{tot}} - \Delta N) \sqrt{\frac{2(U^{\text{tot}} - \Delta U)}{3(N^{\text{tot}} - \Delta N)}} \right. \quad (55)$$

$$\left. - \frac{1}{2} (N^{\text{tot}} + \Delta N) \sqrt{\frac{2(U^{\text{tot}} + \Delta U)}{3(N^{\text{tot}} + \Delta N)}} \right), \quad (56)$$

$$\simeq -\frac{SN^{\text{tot}}}{V\sqrt{3\pi m}} \sqrt{\frac{U^{\text{tot}}}{N^{\text{tot}}}} \left( \frac{\Delta U}{U^{\text{tot}}} + \frac{\Delta N}{N^{\text{tot}}} \right) + \mathcal{O}(\Delta N^2, \Delta U^2). \quad (57)$$

Analogously  $\Delta U$  obeys the following differential equation

$$\frac{d}{dt} \Delta U = \frac{d}{dt} U^A - \frac{d}{dt} U^B, \quad (58)$$

$$= 2(J_E^{B \rightarrow A} - J_E^{A \rightarrow B}), \quad (59)$$

$$= 2 \frac{S}{V} \sqrt{\frac{2k_B}{\pi m}} (N_B k_B T_B \sqrt{T_B} - N_A k_B T_A \sqrt{T_A}), \quad (60)$$

$$\simeq \frac{4SU^{\text{tot}}}{3V} \sqrt{\frac{k_B}{3\pi m}} \sqrt{\frac{U^{\text{tot}}}{N^{\text{tot}}}} \left( -3 \frac{\Delta U}{U^{\text{tot}}} + \frac{\Delta N}{N^{\text{tot}}} \right) + \mathcal{O}(\Delta N^2, \Delta U^2). \quad (61)$$

$$(62)$$

We write the coupled evolution equations

$$\frac{d}{dt} \Delta \bar{N} = -\Gamma (\Delta \bar{U} + \Delta \bar{N}), \quad (63)$$

$$\frac{d}{dt} \Delta \bar{U} = -\frac{4}{3} \Gamma (3\Delta \bar{U} - \Delta \bar{N}), \quad (64)$$

$$(65)$$

where  $\Gamma = \frac{S}{V\sqrt{3\pi m}} \sqrt{\frac{U^{\text{tot}}}{N^{\text{tot}}}}$ .

This system of equation can be written as a matrix vector problem

$$\frac{d}{dt} \begin{pmatrix} \Delta \bar{N} \\ \Delta \bar{U} \end{pmatrix} = -\Gamma \begin{pmatrix} 1 & 1 \\ -\frac{4}{3} & 4 \end{pmatrix} \begin{pmatrix} \Delta \bar{N} \\ \Delta \bar{U} \end{pmatrix}, \quad (66)$$

$$\frac{d}{dt} \begin{pmatrix} \Delta \bar{N} \\ \Delta \bar{U} \end{pmatrix} = -\Gamma M. \begin{pmatrix} \Delta \bar{N} \\ \Delta \bar{U} \end{pmatrix}. \quad (67)$$

The matrix  $M$  admits the following Eigenvalues and their Eigenvectors

$$\lambda_1 = -\Gamma \frac{15 + \sqrt{33}}{6} \quad \vec{v}_1 = \begin{pmatrix} \frac{9 - \sqrt{33}}{8} \\ 1 \end{pmatrix}, \quad (68)$$

$$\lambda_2 = -\Gamma \frac{15 - \sqrt{33}}{6} \quad \vec{v}_2 = \begin{pmatrix} \frac{9 + \sqrt{33}}{8} \\ 1 \end{pmatrix}. \quad (69)$$

$$(70)$$

The general solution of the differential equation is then written

$$\begin{pmatrix} \Delta \bar{N}(t) \\ \Delta \bar{U}(t) \end{pmatrix} = C_1 e^{-\Gamma \lambda_1 t} \vec{v}_1 + C_2 e^{-\Gamma \lambda_2 t} \vec{v}_2, \quad (71)$$

where both relaxation rates  $-\lambda_{1/2} = -\Gamma \frac{15 \pm \sqrt{33}}{6} < 0$  are decaying rates.