Non-equilibrium physics WS 20/21 – Exercise Sheet 9:

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1 Discussion:

i) What is the Knudsen number Kn? How does a Boltzmann gas behave in the limits $Kn \ll 1$ and $Kn \gg 1$?

2 In-class problems:

2.1 Ideal hydrodynamics

Consider a monoatomic Boltzmann gas for which we derived the following balance equations for the mass density $\rho(t,\vec{r}) = m\int_{\vec{p}} f(t,\vec{r},\vec{p})$, the velocity field $\rho(t,\vec{r})\vec{v}(t,\vec{r}) = \int_{\vec{p}} \vec{p} f(t,\vec{r},\vec{p})$ and the internal energy density $e(t,\vec{r}) = \int_{\vec{p}} \frac{[\vec{p}-m\vec{v}(t,\vec{r})]^2}{2m} f(t,\vec{r},\vec{p})$

$$\begin{aligned}
\frac{\partial}{\partial t}\rho(t,\vec{r}) + \vec{\nabla}\Big(\rho(t,\vec{r})\vec{v}(t,\vec{r})\Big) &= 0\ , \\
\frac{\partial}{\partial t}\rho(t,\vec{r})v^i(t,\vec{r}) + \frac{\partial}{\partial r_j}\Big(\rho(t,\vec{r})\vec{v}^i(t,\vec{r})\vec{v}^j(t,\vec{r}) + \Pi^{ij}(t,\vec{r})\Big) &= \frac{\rho(t,\vec{r})}{m}F^i(t,\vec{r})\ , \\
\frac{\partial}{\partial t}e(t,\vec{r}) + \vec{\nabla}\Big(e(t,\vec{r})\vec{v}(t,\vec{r}) + \vec{J}_U(t,\vec{r})\Big) &= -\Pi^{ij}(t,\vec{r})\frac{\partial v^i}{\partial r_j}(t,\vec{r})\ ,
\end{aligned}$$

where the $\Pi^{ij}(t,\vec{r})$ denotes the stress tensor

$$\Pi^{ij}(t,\vec{r}) = \rho(t,\vec{r})\left\langle \frac{[\vec{p}^i - m\vec{v}^i(t,\vec{r})]}{m}\frac{[\vec{p}^j - m\vec{v}^i(t,\vec{r})]}{m}\right\rangle_{\vec{p}} = \int_{\vec{p}} \frac{[\vec{p}^i - m\vec{v}^i(t,\vec{r})][\vec{p}^j - m\vec{v}^j(t,\vec{r})]}{m} f(t,\vec{r},\vec{p})$$

and $\vec{J}_U(t,\vec{r})$ is the energy flux in the local rest frame

$$\vec{J}_U(t,\vec{r}) = \frac{1}{2}\rho(t,\vec{r})\left\langle \Big(\vec{p}/m - \vec{v}(t,\vec{r})\Big)^2[\vec{p}/m - \vec{v}(t,\vec{r})]\right\rangle_{\vec{p}} = \int_{\vec{p}} \frac{\Big(\vec{p} - m\vec{v}(t,\vec{r})\Big)^2}{2m}[\vec{p}/m - \vec{v}(t,\vec{r})] f(t,\vec{r},\vec{p})$$

i) Calculate the stress-tensor $\Pi^{ij,(0)}(t,\vec{r})$ and energy flux $\vec{J}_U^{(0)}(t,\vec{r})$ for a Boltzmann gas in local thermal equilibrium $f(t,\vec{r},\vec{p}) = f^{(0)}(T(t,\vec{r}), n(t,\vec{r}), \vec{v}(t,\vec{r}), \vec{p})$.

ii) Show that for $\Pi^{ij} = \Pi^{ij,(0)}$ and $\vec{J}_U = \vec{J}_U^{(0)}$ the balance equation reduce to the equations of motion for an ideal fluid

3 Homework problems:

3.1 Non-equilibrium corrections to ideal hydrodynamics

Based on the Chapman-Enskog expansion the first non-trivial corrections $f^{(1)}(t,\vec{r},\vec{p})$ to the local equilibrium distribution $f^{(0)}(t,\vec{r},\vec{p}) = n(t,\vec{r})\left(\frac{2\pi\hbar^2}{mk_BT(t,\vec{r})}\right)^{3/2}\exp\left(-\frac{[\vec{p}-m\vec{v}(t,\vec{r})]^2}{2mk_BT(t,\vec{r})}\right)$ are determined according to

$$\left(\frac{\partial}{\partial t} + \frac{\vec{p}}{m}\vec{\nabla}_{\vec{r}} + \vec{F}(t,\vec{r})\vec{\nabla}_{\vec{p}}\right) f^{(0)}(t,\vec{r},\vec{p}) = \delta C[f^{(0)}, f^{(1)}](t,\vec{r},\vec{p}) \;, \tag{1}$$

where $\delta C[f^{(0)}, f^{(1)}]$ denotes the linearized collision operator.

i) Show that in the relaxation time approximation the non-equilibrium correction $f^{(1)}(t,\vec{r},\vec{p})$ is given by

$$\begin{aligned} f^{(1)}(t,\vec{r},\vec{p}) \;&=\; -\tau_R f^0(t,\vec{r},\vec{p})\left[\frac{1}{\rho(t,\vec{r})}\mathcal{D}\rho(t,\vec{r}) + \left(\frac{m\vec{u}_{\vec{p}}^2}{2k_BT(t,\vec{r})} - \frac{3}{2}\right)\frac{1}{T(t,\vec{r})}\mathcal{D}T(t,\vec{r})\right.\\ &\left. + \frac{m\vec{u}_{\vec{p}}^i}{k_BT(t,\vec{r})}\mathcal{D}\vec{v}^i(t,\vec{r}) - \frac{\vec{u}_{\vec{p}}}{k_BT(t,\vec{r})}\vec{F}(t,\vec{r})\right] \end{aligned}$$

where $\mathcal{D} = \frac{\partial}{\partial t} + \frac{\vec{p}}{m}\vec{\nabla}_{\vec{r}}$ and we denote $\vec{u}_{\vec{p}} = \vec{p}/m - \vec{v}(t,\vec{r})$.

ii) Exploit the equations of motions of ideal hydrodynamics

$$\begin{aligned} \left(\frac{\partial}{\partial t} + \vec{v}(t,\vec{r})\vec{\nabla}\right)\rho(t,\vec{r}) \;&=\; -\rho(t,\vec{r})\;\vec{\nabla}\vec{v}(t,\vec{r}) \;,\\ \frac{\partial}{\partial t}\vec{v}^i(t,\vec{r}) + \vec{v}^j(t,\vec{r})\frac{\partial}{\partial r_j}\vec{v}^i(t,\vec{r}) \;&=\; -\frac{1}{\rho(t,\vec{r})}\frac{\partial}{\partial r_i}P(t,\vec{r}) + \frac{1}{m}\vec{F}^i(t,\vec{r}) \;,\\ \frac{\partial}{\partial t}e(t,\vec{r}) + \vec{\nabla}\Big(e(t,\vec{r})\vec{v}(t,\vec{r})\Big) \;&=\; -P(t,\vec{r})\;\vec{\nabla}\vec{v}(t,\vec{r}) \;, \end{aligned}$$

to show that the expression for $f^{(1)}(t,\vec{r},\vec{p})$ can be compactly expressed as

$$\begin{aligned} f^{(1)}(t,\vec{r},\vec{p}) \;&=\; -\tau_R f^0(t,\vec{r},\vec{p})\left[\left(\frac{m\vec{u}_{\vec{p}}^2}{2k_BT(t,\vec{r})} - \frac{5}{2}\right)\vec{u}_{\vec{p}}\frac{\vec{\nabla}T(t,\vec{r})}{T(t,\vec{r})}\right.\\ &\left. + \frac{m}{2k_BT(t,\vec{r})}\left(\frac{\partial v_i}{\partial r_j}(t,\vec{r}) + \frac{\partial v_j}{\partial r_i}(t,\vec{r})\right)\left(\vec{u}_{\vec{p}}^i\vec{u}_{\vec{p}}^j - \frac{1}{3}\delta^{ij}\vec{u}_{\vec{p}}^2\right)\right] \end{aligned}$$

(Hint: Express the energy density and pressure in terms of the temperatures and densities, by using the equations of state $e(t,\vec{r}) = \frac{3}{2}n(t,\vec{r})k_BT(t,\vec{r})$ and $P(t,\vec{r}) = n(t,\vec{r})k_BT(t,\vec{r})$ to derive the evolution equation for the temperature $T(t,\vec{r})$

$$\left(\frac{\partial}{\partial t} + \vec{v}(t,\vec{r})\vec{\nabla}\right)T(t,\vec{r}) \;=\; -\frac{2}{3}T(t,\vec{r})\;\vec{\nabla}\vec{v}(t,\vec{r}) \;, \tag{2}$$

Eliminate all time derivatives in the expression for $f^{(1)}$ in favor or spatial derivatives using the ideal equations of motion.)

iii) Verify explicitly that $f^{(1)}(t,\vec{r},\vec{p})$ does not contribute to the quantities $\rho(t,\vec{r})$, $v(t,\vec{r})$ and $e(t,\vec{r})$. (Hint: Change integration variables to $d^3\vec{u}_{\vec{p}}$ and consider the symmetries of the integrand)

iv) Calculate the heat conductivity κ in the constitutive relation $\vec{J}_U = -\kappa\vec{\nabla}T(t,\vec{r})$ (Hint: $\int_{\vec{p}}\frac{(m\vec{u}_{\vec{p}})^2}{2m}(m\vec{u}_{\vec{p}})^2\left(\frac{(m\vec{u}_{\vec{p}})^2}{2mk_BT} - \frac{5}{2}\right)f^{(0)} = \frac{15}{2}mn(k_BT)^2$)