Non-equilibrium physics WS 20/21 – Exercise Sheet 8:

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1 Discussion:

i) Discuss the definition, physical meaning and properties of entropy in classical and quantum statistical system, and in the context of the Boltzmann equation.

ii) What is the form of the stationary solution of the Boltzmann equation for a homogenous system in the absence of external forces? What is the difference between the concepts of *balance* and *detailed balance*? What is the *relaxation time approximation*?

2 In-class problems:

2.1 Global equilibrium in the presence of a scalar potential

Consider a dilute gas of particles, whose dynamics is described by the Boltzmann equation, in the presence of an external force $\vec{F}(\vec{r}) = -\vec{\nabla}_{\vec{r}}V(\vec{r})$ derived from a scalar potential $V(\vec{r})$

i) Show that the global equilibrium solution is of the form $f_{\text{eq}}(\vec{r},\vec{p}) = n(\vec{r})\left(\frac{2\pi\hbar^2}{mk_BT}\right)^{3/2} e^{-\frac{\vec{p}^2}{2mk_BT}}$ and determine the spatial profile of density $n(\vec{r})$

3 Homework problems:

3.1 Boltzmann gas in a harmonic trap

Consider a dilute gas of particles described by the Boltzmann equation

$$\left(\frac{\partial}{\partial t} + \frac{\vec{p}}{m}\vec{\nabla}_{\vec{r}} + \vec{F}(\vec{r})\vec{\nabla}_{\vec{p}}\right) f(t,\vec{r},\vec{p}) = C[f](t,\vec{r},\vec{p})\ , \tag{1}$$

in the presence of an external force $\vec{F}(\vec{r}) = -\vec{\nabla}_{\vec{r}}V(\vec{r})$ derived from a harmonic potential

$$V(\vec{r}) = \frac{1}{2}m\omega^2\vec{r}^2.$$

i) Determine the global equilibrium solution $f_{\text{eq}}(\vec{r},\vec{p})$ for this system.

ii) Show that for a generic function $g(\vec{r},\vec{p})$ of coordinates and momenta, the evolution of the average of this quantity

$$\langle g(\vec{r},\vec{p})\rangle \equiv \int \frac{d^3\vec{r}d^3\vec{p}}{(2\pi\hbar)^3} f(t,\vec{r},\vec{p})\ g(\vec{r},\vec{p})$$

is governed by

$$\frac{d\langle g(\vec{r},\vec{p})\rangle}{dt} - \left\langle \frac{\vec{p}}{m}\,\vec{\nabla}_{\vec{r}} g(\vec{r},\vec{p})\right\rangle - \left\langle \vec{F}(\vec{r})\,\vec{\nabla}_{\vec{p}} g(\vec{r},\vec{p})\right\rangle = \int \frac{d^3\vec{r}d^3\vec{p}}{(2\pi\hbar)^3}\, g(\vec{r},\vec{p})\, C[f](t,\vec{r},\vec{p}) \tag{2}$$

iii) Explain why for $g(\vec{r},\vec{p}) = g_N(\vec{r}) + \vec{g}_{\vec{p}}(\vec{r})\vec{p} + g_e(\vec{r})\frac{\vec{p}^2}{2m}$ the right hand side of Eq. (2) vanishes irrespective of the spatial dependence of the coefficient functions $g_{N,\vec{p},e}(\vec{r})$.

iv) Derive the explicit form for the equations of motion for the quantities $e_{\rm pot}(\vec{r},\vec{p}) = \frac{1}{2}m\omega^2\vec{r}^2$, $e_{\rm kin}(\vec{r},\vec{p}) = \frac{\vec{p}^2}{2m}$ and $e_{\rm corr}(\vec{r},\vec{p}) = \omega\frac{\vec{r}\cdot\vec{p}}{2}$.

v) Based on your results in (iv) show that the Boltzmann gas in a harmonic trap can exhibit oscillatory behavior in the long time limit, and therefore does not relax towards the global equilibrium solution. Determine the frequency of oscillations.

3.2 Electric conductivity & eff. relaxation time of a Lorentz gas

Consider a dilute gas of light particles of mass m and heavy particles of mass M, dominated by elastic interactions between light and heavy particles. Since the kinetic motion of heavy particles is suppressed by their large mass, they can be described as static scattering centers; the dynamics of the light particles is then governed by the kinetic equation for a Lorentz gas

$$\left(\frac{\partial}{\partial t} + \frac{\vec{p}}{m}\vec{\nabla}_{\vec{r}} + \vec{F}\vec{\nabla}_{\vec{p}}\right) f_{\rm light}(t,\vec{r},\vec{p}) = C[f_{\rm light}](t,\vec{r},\vec{p}) \tag{3}$$

$$C[f_{\rm light}](t,\vec{r},\vec{p}) = n_{\rm heavy}\frac{|\vec{p}|}{m}\int d\Omega_{\vec{p}\vec{p}'}\frac{d\sigma}{d\Omega_{\vec{p}\vec{p}'}}(\vec{p}\to\vec{p}')\left[f_{\rm light}(t,\vec{r},\vec{p}') - f_{\rm light}(t,\vec{r},\vec{p})\right] \tag{4}$$

where $n_{\rm heavy}$ denotes the (uniform) density of heavy particles in the system, $\Omega_{\vec{p},\vec{p}'}$ is the scattering angle and $\frac{d\sigma}{d\Omega}(\vec{p}\to\vec{p}')$ denotes the cross-section for the interaction.

We will assume that the interactions are elastic and particle number conserving, i.e. the number of light particles is conserved and the energy $\Delta E = \frac{(\vec{p}-\vec{p}')^2}{2M}$ transferred to the heavy particles is negligible. Nevertheless, momentum can be transferred from light to heavy particles, i.e. the differential cross-section $\frac{d\sigma}{d\Omega}(\vec{p}\to\vec{p}')$ is non-zero even when $\vec{p}\neq\vec{p}'$.

i) Show that local equilibrium solutions for $f = f_{\rm light}$ are of the form

$$f^{(0)}(t,\vec{r},\vec{p}) = \exp\left(-\frac{\epsilon_{\vec{p}} - \mu(t,\vec{r})}{k_B T(t,\vec{r})}\right), \qquad \epsilon_{\vec{p}} = \vec{p}^2/2m\ . \tag{5}$$

What differences do you observe in comparison to local equilibrium solutions of the Boltzmann equation for two-body interactions between light particles?

We will assume in the following that the differential cross section $\frac{d\sigma}{d\Omega}(\vec{p}\to\vec{p}')$ is a function of of the magnitude of the momentum $|\vec{p}| = |\vec{p}'|$ and the scattering angle $\theta_{pp'}$ only. We now consider the effect of a constant external electric field $\vec{E}$ in the limit where the change in velocity due to the Lorentz force between individual collisions is small compared to the thermal velocity $\frac{q|\vec{E}|}{m}\tau_{\rm mfp} \ll v_{\rm th}$.

ii) Demonstrate that to leading order in $\frac{q|\vec{E}|}{mv_{\rm th}}\tau_{\rm mfp} \ll 1$, the stationary solutions to the Boltzmann equation for a spatially homoegenous Lorentz gas are given by $f(\vec{p}) = f^{(0)}(\vec{p}) + f^{(1)}(\vec{p})$, where $f^{(0)}(\vec{p})$ is the local equilibrium distribution and $f^{(1)}(\vec{p})$ is determined by

$$\delta C[f^{(1)}](\vec{p}) = -q\frac{\vec{p}\cdot\vec{E}}{mk_BT}f^{(0)}(\vec{p})\ , \tag{6}$$

where

$$\delta C[f^{(1)}](\vec{p}) = n_{\text{heavy}} \frac{|\vec{p}|}{m} \int d\Omega_{\vec{p}\vec{p}'} \frac{d\sigma}{d\Omega_{\vec{p}\vec{p}'}} (\vec{p} \to \vec{p}') \left[f^{(1)}(\vec{p}') - f^{(1)}(\vec{p}) \right] \tag{7}$$

Based on the relaxation time approximation (RTA), the linearized collision operator $\delta C[f^{(1)}]$ is approximated as

$$\delta C[f^{(1)}](\vec{p}) \Big|_{RTA} = -\frac{f^{(1)}(\vec{p})}{\tau_R(\epsilon_{\vec{p}})} \tag{8}$$

with an energy dependent relaxation time $\tau_R(\epsilon_{\vec{p}})$ and the functional form of the distribution $f^{(1)}$ is given by

$$f^{(1)}(\vec{p}) \Big|_{RTA} = q\tau_R(\epsilon_{\vec{p}}) \frac{\vec{p} \cdot \vec{E}}{mk_B T} f^{(0)}(\vec{p}) \ . \tag{9}$$

iii) Show that by equating δC in (7) and (8) and using the solution in (9) for the functional form of the distribution $f^{(1)}$ the energy dependent relaxation time $\tau_R(\epsilon_{\vec{p}})$ can be determined self-consistently according to

$$\frac{1}{\tau_R(\epsilon_{\vec{p}})} = 2\pi n_{\text{heavy}} \frac{|\vec{p}|}{m} \int d\cos(\theta_{pp'}) \frac{d\sigma}{d\Omega_{pp'}} \left[1 - \cos(\theta_{pp'}) \right] \tag{10}$$

$\Big($Hint: By appropriate choice of coordinates you can express $\vec{p} \cdot \vec{E} = |\vec{p}||\vec{E}| \cos(\theta_p)$ and $\vec{p}' \cdot \vec{E} = |\vec{p}'||\vec{E}| \cos(\theta_p) \cos(\theta_{pp'}) + \sin(\theta_p) \sin(\theta_{pp'}) \cos(\phi_{p'})$ $\Big)$

iv) Determine the energy dependent relaxation time $\tau_R(\epsilon_{\vec{p}})$ for the scattering off hard-sphere scattering centers $\frac{d\sigma}{d\Omega_{pp'}} = \frac{R^2}{4}$ and calculate the electrical conductivity σ_{el} for this model.