Non-equilibrium physics WS 20/21 – Exercise Sheet 6:

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1 Discussion:

i) What is a *mean-field approximation* and under what conditions is it applicable to a systems of interacting particles? What is the physical meaning of the different terms in the Boltzmann equation? What are the assumptions underlying the derivation of the Boltzmann equations?

2 In-class problems:

2.1 Free-streaming

Consider a *free-streaming* system of particles, which is described by the collisionless Boltzmann equation in the absence of external forces.

i) Show that the general solution for the single particle distribution of a free-streaming system is of the form $f(t, \vec{r}, \vec{p}) = f\left(t_0, \vec{r} - \frac{\vec{p}}{m}(t - t_0), \vec{p}\right)$

2.2 Constant force

Consider a system of non-interacting particles, which is described by the collisionless Boltzmann equation in the presence of a constant external force $\vec{F}(t, \vec{r}) = \vec{F}_0$.

i) Determine the general solution for the single particle distribution $f(t, \vec{r}, \vec{p})$ for a general initial condition $f(t_0, \vec{r}, \vec{p}) = f_0(\vec{r}, \vec{p})$.

3 Homework problems:

3.1 AC conductivity of a collisionless plasma

We consider a plasma of a single species of charged particles, described by the collisionless Boltzmann equation in position ($\vec{r}$) and velocity ($\vec{v}=\frac{\vec{p}_{\rm kin}}{m}$) space

$$\left[\frac{\partial}{\partial t}+\vec{v}\cdot\vec{\nabla}_{\vec{r}}+\frac{e}{m}\left(\vec{E}(t,\vec{r})+\frac{\vec{v}}{c}\times\vec{B}(t,\vec{r})\right)\vec{\nabla}_{\vec{v}}\right]f(t,\vec{r},\vec{v})=0\;, \tag{1}$$

in the presence of a small external electric field

$$\vec{E}(t,\vec{r})=\lambda\vec{E}_0e^{i[\vec{k}\vec{r}-\omega t]}\;.$$

and no magnetic field $\vec{B}(t,\vec{r})$.

i) Develop a microscopic picture of the motion of the charged particles over a time scale $\Delta t_\omega=\frac{2\pi}{\omega}$. Discuss how the collisionless approximation can be justified in the limit $v_{\rm th}\Delta t_\omega\ll l_{\rm mfp}$ and why the approximation is not suitable to compute the response to a static electric field ($\omega=0$).

Since the general solution to this problem is hard to find, we will instead construct the solution perturbatively, by expanding the single particle distribution according to

$$f(t,\vec{r},\vec{v})=f_0(\vec{v})+\lambda\delta f_{\vec{k},\omega}(t,\vec{r},\vec{v})$$

where $\delta f_{\vec{k},\omega}(t,\vec{r},\vec{v})\ll f_0(\vec{v})$ is a small perturbation generated in response to the external electric field. We consider as our expansion point, an equilibrium distribution of the form

$$f_0(\vec{v})=n_0e^{\frac{-m\vec{v}^2}{2k_BT}}\left(\frac{2\pi\hbar^2}{mk_BT}\right)^{3/2}\;,$$

where n_0 denotes the local particle number density.

ii) Show that the background distribution f_0 satisfies the evolution equation $\left[\frac{\partial}{\partial t}+\vec{v}\cdot\vec{\nabla}_{\vec{r}}\right]f_0=0$. Construct the evolution equation for the perturbations $\delta f_{\vec{k},\omega}(t,\vec{r},\vec{v})$ by collecting all residual terms of $O(\lambda)$ in Eq. (1), neglecting terms of order $O(\lambda^2)$ and higher.

Since in the long time limit, the space-time dependence of the perturbation is expected to follow that of the external electric field, we will search for solutions of the form

$$\delta f_{\vec{k},\omega}(t,\vec{r},\vec{v})=\delta f_{\vec{k},\omega}(\vec{v})e^{i[\vec{k}\vec{r}-(\omega+i\epsilon)t]}\;,$$

where in the above expression $\epsilon>0$ is inserted to ensure that $\delta f_{\vec{k},\omega}(\vec{v})\to 0$ in the limit $t\to-\infty$, and we will take the limit $\epsilon\to 0$ in the final step of our calculation.

iii) Show that the solution for $\delta f_{\vec{k},\omega}(\vec{v})$ can be expressed as

$$\delta f_{\vec{k},\omega}(\vec{v})=\frac{ie}{k_BT}\frac{\vec{E}_0\cdot\vec{v}}{\omega+i\epsilon-\vec{v}\cdot\vec{k}}f_0(\vec{v}) \tag{2}$$

Based on the solution in Eq. (2) we will now proceed to calculate the conductivity tensor $\sigma(\vec{k},\omega)$, which according to $J^i=\sigma^{ij}E^j$ relates the induced current

$$J^i=e\int\frac{m^3d^3\vec{v}}{(2\pi\hbar)^3}\;v^i\;f(t,\vec{r},\vec{v})$$

to the external electric field $\vec{E}$.

iv) Show the components of the conductivity can be expressed as

$$\sigma^{ij} = \frac{ie^2 m^3}{k_B T} \int \frac{d^3\vec{v}}{(2\pi\hbar)^3} \frac{\vec{v}^i \ \vec{v}^j}{\omega + i\epsilon - \vec{v}\cdot\vec{k}} \ f_0(\vec{v})$$

v) Show that the conductivity tensor σ is of the diagonal form $\sigma = diag(\sigma_T, \sigma_T, \sigma_L)$ where $\sigma_{T/L}$ denote the transverse (T) and longitudinal components (L) w.r.t. the wave-vector $\vec{k}$.

vi) Calculate the real and imaginary part of the transverse conductivity $\sigma_T(\vec{k}, \omega)$

vii) Calculate the real and imaginary part of the longitudinal conductivity $\sigma_L(\vec{k}, \omega)$

Some hints on the evaluation of integrals

a) Separate the integral $\int d^3\vec{v}$ into integrations over the longitudinal $v_z = \frac{\vec{k}\cdot\vec{v}}{|\vec{k}|}$ and transverse velocities and first perform the integration over the transverse velocities.

b) Express the integral over the longitudinal velocity v_z in the form

$$\lim_{\epsilon\to 0} \int_{-\infty}^{\infty} dx \ \frac{f(x)}{x + i\epsilon}$$

which can be evaluated as

$$\lim_{\epsilon\to 0} \int_{-\infty}^{\infty} dx \ \frac{f(x)}{x + i\epsilon} = -i\pi f(0) + \mathrm{P} \int_{-\infty}^{\infty} dx \ \frac{f(x)}{x}$$

where the principle value (P) of the integral can be calculated according to

$$\mathrm{P} \int_{-\infty}^{\infty} dx \ \frac{f(x)}{x} = \int_{0}^{\infty} dx \ \frac{f(x) - f(-x)}{x} \ .$$

c) Some useful formulae for the remaining integrals include ($\sigma > 0$)

$$\begin{aligned}
\frac{1}{\sqrt{2\pi\sigma^2}} \int_0^\infty dx \ \left(\frac{x}{\sigma}\right) \ \sinh(ax/\sigma^2) \ e^{-\frac{x^2}{2\sigma^2}} &= \frac{a}{2\sigma} e^{\frac{a^2}{2\sigma^2}} \ , \\
\frac{1}{\sqrt{2\pi\sigma^2}} \int_0^\infty dx \ \frac{\sinh(ax/\sigma^2)}{x/\sigma} \ e^{-\frac{x^2}{2\sigma^2}} &= \frac{1}{2}\sqrt{\frac{\pi}{2}} Erfi\left(\frac{a}{\sqrt{2}\sigma}\right) \ , \\
\frac{1}{\sqrt{2\pi\sigma^2}} \int_0^\infty dx \ \cosh(ax/\sigma^2) \ e^{-\frac{x^2}{2\sigma^2}} &= \frac{1}{2} e^{\frac{a^2}{2\sigma^2}} \ ,
\end{aligned} \tag{3}$$