Non-equilibrium physics WS 20/21 – Exercise Sheet 3:

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1 Discussion:

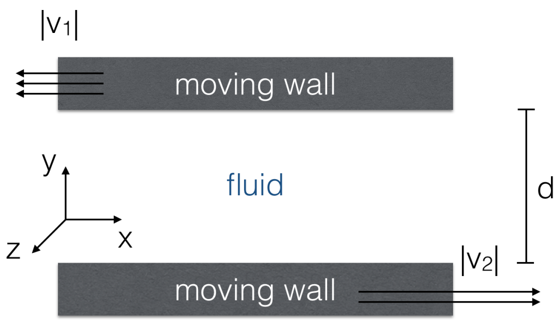
i) What is the physical relvance of the different terms in the *Navier Stokes equation*?

$$\rho\left[\frac{\partial\vec{v}}{\partial t}+(\vec{v}\cdot\vec{\nabla})\vec{v}\right]=-\vec{\nabla}P+\zeta\vec{\nabla}\left(\vec{\nabla}\cdot\vec{v}\right)+\eta\left(\triangle\vec{v}+\frac{1}{3}\vec{\nabla}\left(\vec{\nabla}\cdot\vec{v}\right)\right) \tag{1}$$

2 In-class problems:

2.1 Shear-flow between two parallel moving plates

Consider an *incompressible* fluid, described by the Navier-Stokes equation with constant transport coefficients ζ, η. The fluid is confined between two parallel plates of infinite extent, separated by a distance d in the y direction and moving with different velocities velocities $\vec{v}_1 = (-|v_1|, 0, 0)$ and $\vec{v}_2 = (|v_2|, 0, 0)$ in the $\pm x$-direction.



i) Write down the equations of motion for the non-vanishing components of the fluid velocity $\vec{v}$.

ii) Determine the stationary velocity profile $\vec{v}(x, y, z)$ of the fluid.

3 Homework problems:

3.1 Sound & shear waves

Consider a compressible fluid of constant density $\rho(t,\vec{x}) = \rho_0$, with an equation of state $p = c_s^2 \rho$ which is at rest $\vec{v}(t,\vec{x}) = 0$ in the laboratory frame. We will consider the propagation of density $\delta\rho$ and velocity perturbations δv, whose evolution is described by the linearized continuity equation and the linearized Navier-Stokes equation. We first consider so called sound waves, which are perturbations of the form

$$\delta\rho(t,\vec{x}) = \delta\rho_{\vec{k}}(t) e^{i\vec{k}\vec{x}} \,, \qquad \delta\vec{v}(t,\vec{x}) = \frac{\vec{k}}{|\vec{k}|}\delta v_{\vec{k},\parallel}(t) e^{i\vec{k}\vec{x}}$$

i) Derive the linearized evolution equations for sound waves.

ii) Show that the general solution for sound waves can be expressed in the form

$$\delta\rho_{\vec{k}}(t) = \sum_{\pm} c_{\vec{k}}^{\pm} \;\; \rho_0 \;\; e^{i\omega_{\pm}(\vec{k})t} \,, \tag{2}$$

$$\delta v_{\vec{k},\parallel}(t) = \sum_{\pm} c_{\vec{k}}^{\pm} \left(\frac{-\omega_{\pm}(\vec{k})}{|\vec{k}|} \right) e^{i\omega_{\pm}(\vec{k})t} \,, \tag{3}$$

and determine the dispersion relation $\omega_{\pm}(\vec{k})$. What differences do you observe between ideal ($\eta = \zeta = 0$) and viscous fluid dynamics?

Next we will consider shear waves, which are velocity perturbations of the form

$$\delta\vec{v}(t,\vec{x}) = \delta\vec{v}_{\vec{k},\perp}(t) e^{i\vec{k}\vec{x}} \tag{4}$$

which are transverse to the wave-vector $\vec{k}$, meaning that $\delta\vec{k} \cdot \vec{v}_{\vec{k},\perp}(t) = 0$.

iii) Show with the help of the linearized continuity equation that shear waves do not induce density perturbations, i.e. that they are consistent with $\delta\rho = 0$.

iv) Derive the linearized evolution equation for shear waves. Show that the solution can be expressed in the form

$$\delta\vec{v}_{\vec{k},\perp}(t) = \vec{c}_{\vec{k}}^{\perp} e^{+i\omega(\vec{k})t} \,, \tag{5}$$

and determine the dispersion relation $\omega(\vec{k})$. What differences do you observe between shear waves and sound waves?

3.2 Flow in a donut pipe

Consider the flow of an incompressible fluid in a "donut shaped" pipe of length l, with inner and outer radii R_1 and R_2, as illustrated in the figure.

i) Determine the velocity profile $v_x(r)$ for stationary transport, for a given pressure difference Δp between the two ends of the pipe ($r = \sqrt{y^2 + z^2}$).

ii) Calculate the rate of mass transport (or discharge) $Q = \int d^2\vec{\sigma}\ (\rho\vec{v})$ for the donut pipe.