Non-equilibrium physics WS 20/21 – Exercise Sheet 10:

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1 Discussion:

i) What is *stationary turbulence*? What is the *inertial range* of a turbulent cascade?

2 In-class problems:

2.1 Collinear branching cascade

Consider a one-dimensional branching process, where particles with energy $\omega(k) = ck$ can split into fragments with energies $z\omega(k)$ and $(1-z)\omega(k)$ with a rate $\Gamma(k,z)$. Neglecting the inverse process, the dynamics of this branching process is described by a kinetic equation

$$\partial_t n(k,t) = I[n](k,t) \tag{1}$$

for the particle number density $n(k,t)$, with the collision kernel given by

$$I[n](k,t) = \int_0^1 dz \; \left[\Gamma\left(\frac{k}{z},z\right)\frac{\omega\left(\frac{k}{z}\right)}{\omega(k)} n\left(\frac{k}{z},t\right) + \Gamma\left(\frac{k}{1-z},z\right)\frac{\omega\left(\frac{k}{1-z}\right)}{\omega(k)} n\left(\frac{k}{1-z},t\right) - \Gamma(k,z)n(k,t)\right] \; , \tag{2}$$

i) Determine the spectral index s_0 of the stationary Kolmogorov solutions $n(k) = n_0\left(\frac{k_0}{k}\right)^{s_0}$ for a scale invariant splitting rate of the form $\Gamma(k,z) = \gamma_0 \; K(z)\left(\frac{k_0}{k}\right)^m$.

3 Homework problems:

3.1 Kolmogorov-Zakharov spectra for four wave interactions

Consider the four wave kinetic equation

$$\partial_t n(k,t) = I[n](k,t) \; ,$$

for a statistically homogenous and isotropic system of waves, with power law dispersion $\omega(k) = \omega_0(k/k_0)^z$ with the collision integral given by

$$I[n](k,t) = \frac{1}{2\Omega^{(d)}}\int d\Omega_k d\Omega_{k_1} d\Omega_{k_2} d\Omega_{k_3} \int dk_1 k_1^{d-1} \int dk_2 k_2^{d-1} \int dk_3 k_3^{d-1}$$
$$\times \widetilde{w}(kk_1 \to k_2k_3) n(k,t)n(k_1,t)n(k_2,t)n(k_3,t)\left[\frac{1}{n(k,t)} + \frac{1}{n(k_1,t)} - \frac{1}{n(k_2,t)} - \frac{1}{n(k_3,t)}\right]$$

where

$$\widetilde{w}(kk_1 \to k_2k_3) = (2\pi)\delta(\omega(k)+\omega(k_1)-\omega(k_2)-\omega(k_3))\delta^{(d)}(k+k_1-k_2-k_3)\frac{|T(k,k_1,k_2,k_3)|^2}{2} \quad (3)$$

for a scale invariant matrix element $T(\lambda k, \lambda k_1, \lambda k_2, \lambda k_3) = \lambda^m T(k,k_1,k_2,k_3)$ with the usual symmetry properties $T(k,k_1,k_2,k_3) = T(k_1,k,k_3,k_2) = T(k_2,k_3,k,k_1)$.

i) Show that for a scale invariant spectrum of the form $n(k) = n_0\left(\frac{k_0}{k}\right)^{s_0}$, the collision integral can be re-expressed in the following way by performing appropriate Zakharov transformations

$$I[n](k,t) = \frac{1}{2\Omega^{(d)}}\int d\Omega_k d\Omega_{k_1} d\Omega_{k_2} d\Omega_{k_3} \int dk_1 k_1^{d-1} \int dk_2 k_2^{d-1} \int dk_3 k_3^{d-1}$$
$$\times \widetilde{w}(kk_1 \to k_2k_3)\, n(k_1,t)\, n(k_2,t)\, n(k_3,t)\left[1+\left(\frac{k}{k_1}\right)^x - \left(\frac{k}{k_2}\right)^x - \left(\frac{k}{k_3}\right)^x\right]$$

$\Big($Cross-check: $x = 2m - 3s_0 - d - z + 4d$ $\Big)$

ii) Determine the scaling exponent s_0 for the stationary Kolmogorov solutions of the four wave kinetic equation. How many solutions can you find? Which conserved quantity is transported in the turbulent cascade for the different cases?

3.2 Formulae:

$$x^{a+\epsilon} \simeq x^a + \epsilon x^a \log(x) + \mathcal{O}(\epsilon^2)\ ,$$

$$\int_0^1 dz[(1-z)\log(1-z) + z\log(z)] = -\frac{1}{2}\ ,$$
$$\int_0^1 dz[(1-z)\log(1-z) + z\log(z)]\frac{8}{\pi}\sqrt{z(1-z)} = \frac{5}{6} - 2\log(2) \approx -0.552961$$
$$\int_0^1 dz[(1-z)\log(1-z) + z\log(z)]\frac{1}{\pi\sqrt{z(1-z)}} = 1 - \log(4) \approx -0.386294$$

3.3 Collinear branching cascade (continued)

Consider again the one-dimensional branching process in Eqns. (1,2).

i) Show that for $n(k) = n_0\left(\frac{k_0}{k}\right)^{s_0}$ with $s_0 > 2-m$ the energy flux $\dot{E}(\Lambda) = \int_\Lambda^\infty dk\ \omega(k)\ I[n](k,t)$ can be expressed as

$$\dot{E}(\Lambda) = \gamma_0\ n_0 k_0\ ck_0\ \left(\frac{k_0}{\Lambda}\right)^{m+s_0-2}\int_0^1 dz\ \frac{z^{m+s_0-1} + (1-z)^{m+s_0-1} - 1}{m+s_0-2}\ K(z)\ ,$$

ii) Evaluate the dimensionless integral over z using L'Hospital's rule to take the limit $\epsilon = m+s_0-2$ going to zero, where the spectrum $n(k)$ approaches the stationary Kolmogorov solution. Show that in this limit the stationary flux becomes scale (Λ) independent and is expliclity given by

$$\dot{E}(\Lambda) = \gamma_0\ n_0 k_0\ ck_0\int_0^1 dz\ \ [z\log(z) + (1-z)\log(1-z)]\, K(z) \quad (4)$$

What can you say about the sign of $\dot{E}(\Lambda)$ (assuming that the splitting kernel $K(z)$ is positive semi-definite), and what does this tell you about the direction of the cascade?

iii) Evaluate the stationary energy flux $\dot{E}(\Lambda)$ in Eq. (3) for the splitting kernels

$$K(z) = 1\ , \qquad K(z) = \frac{1}{\pi\sqrt{z(1-z)}}\ , \qquad \text{and} \qquad K(z) = \frac{8}{\pi}\sqrt{z(1-z)} \tag{5}$$

Which kernel is most efficient in transporting energy? Why? $\Big($Note that all of the above kernels are normalized such that the integrated splitting rate $\int_0^1 dz\ \Gamma(k,z) = \gamma_0\ \left(\frac{k_0}{k}\right)^m$ is identical.$\Big)$

We will now focus on the case of a constant splitting kernel $K(z) = 1$ and consider the specific situation where energy is injected into the system with a constant rate $\dot{E}_{\rm in}$ at a characteristic momentum scale $k_{\rm in}$, and removed from the system with the same rate $\dot{E}_{\rm out} = \dot{E}_{\rm in}$ at a much smaller momentum scale $k_{\rm out} \ll k_{\rm in}$.

iv) Determine the exact form of the stationary Kolmogorov solution $n(k)$ (including the non-universal amplitude n_0) in the inertial range of momenta $k_{\rm out} \ll k \ll k_{\rm in}$.