

# Non-equilibrium physics WS 18/19 – Exercise Sheet 9:

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## 1 Discussion:

- i) What is the Knudsen number  $Kn$ ? How does a Boltzmann gas behave in the limits  $Kn \ll 1$  and  $Kn \gg 1$ ?

## 2 In-class problems:

### 2.1 Ideal hydrodynamics

Consider a monoatomic Boltzmann gas for which we derived the following balance equations for the mass density  $\rho(t, \vec{r}) = m \int_{\vec{p}} f(t, \vec{r}, \vec{p})$ , the velocity field  $\rho(t, \vec{r}) \vec{v}(t, \vec{r}) = \int_{\vec{p}} \vec{p} f(t, \vec{r}, \vec{p})$  and the internal energy density  $e(t, \vec{r}) = \int_{\vec{p}} \frac{[\vec{p} - m\vec{v}(t, \vec{r})]^2}{2m} f(t, \vec{r}, \vec{p})$

$$\begin{aligned} \frac{\partial}{\partial t} \rho(t, \vec{r}) + \vec{\nabla} \cdot (\rho(t, \vec{r}) \vec{v}(t, \vec{r})) &= 0, \\ \frac{\partial}{\partial t} \rho(t, \vec{r}) v^i(t, \vec{r}) + \frac{\partial}{\partial r_j} (\rho(t, \vec{r}) \vec{v}^i(t, \vec{r}) v^j(t, \vec{r}) + \Pi^{ij}(t, \vec{r})) &= \frac{\rho(t, \vec{r})}{m} F^i(t, \vec{r}), \\ \frac{\partial}{\partial t} e(t, \vec{r}) + \vec{\nabla} \cdot (e(t, \vec{r}) \vec{v}(t, \vec{r}) + \vec{J}_U(t, \vec{r})) &= -\Pi^{ij}(t, \vec{r}) \frac{\partial v^i}{\partial r_j}(t, \vec{r}), \end{aligned}$$

where the  $\Pi^{ij}(t, \vec{r})$  denotes the stress tensor

$$\Pi^{ij}(t, \vec{r}) = \rho(t, \vec{r}) \left\langle \frac{[\vec{p}^i - m\vec{v}^i(t, \vec{r})]}{m} \frac{[\vec{p}^j - m\vec{v}^j(t, \vec{r})]}{m} \right\rangle_{\vec{p}} = \int_{\vec{p}} \frac{[\vec{p}^i - m\vec{v}^i(t, \vec{r})][\vec{p}^j - m\vec{v}^j(t, \vec{r})]}{m} f(t, \vec{r}, \vec{p})$$

and  $\vec{J}_U(t, \vec{r})$  is the energy flux in the local rest frame

$$\vec{J}_U(t, \vec{r}) = \frac{1}{2} \rho(t, \vec{r}) \left\langle \left( \vec{p}/m - \vec{v}(t, \vec{r}) \right)^2 [\vec{p}/m - \vec{v}(t, \vec{r})] \right\rangle_{\vec{p}} = \int_{\vec{p}} \frac{(\vec{p} - m\vec{v}(t, \vec{r}))^2}{2m} [\vec{p}/m - \vec{v}(t, \vec{r})] f(t, \vec{r}, \vec{p})$$

- i) Calculate the stress-tensor  $\Pi^{ij,(0)}(t, \vec{r})$  and energy flux  $\vec{J}_U^{(0)}(t, \vec{r})$  for a Boltzmann gas in local thermal equilibrium  $f(t, \vec{r}, \vec{p}) = f^{(0)}(T(t, \vec{r}), n(t, \vec{r}), \vec{v}(t, \vec{r}), \vec{p})$ .
- ii) Show that for  $\Pi^{ij} = \Pi^{ij,(0)}$  and  $\vec{J}_U = \vec{J}_U^{(0)}$  the balance equation reduce to the equations of motion for an ideal fluid

### 3 Homework problems:

#### 3.1 Non-equilibrium corrections to ideal hydrodynamics

Based on the Chapman-Enskog expansion the first non-trivial corrections  $f^{(1)}(t, \vec{r}, \vec{p})$  to the local equilibrium distribution  $f^{(0)}(t, \vec{r}, \vec{p}) = n(t, \vec{r}) \left( \frac{2\pi\hbar^2}{mk_B T(t, \vec{r})} \right)^{3/2} \exp\left(-\frac{[\vec{p}-m\vec{v}(t, \vec{r})]^2}{2mk_B T(t, \vec{r})}\right)$  are determined according to

$$\left( \frac{\partial}{\partial t} + \frac{\vec{p}}{m} \vec{\nabla}_{\vec{r}} + \vec{F}(t, \vec{r}) \vec{\nabla}_{\vec{p}} \right) f^{(0)}(t, \vec{r}, \vec{p}) = \delta C[f^{(0)}, f^{(1)}](t, \vec{r}, \vec{p}), \quad (1)$$

where  $\delta C[f^{(0)}, f^{(1)}]$  denotes the linearized collision operator.

- i) Show that in the relaxation time approximation the non-equilibrium correction  $f^{(1)}(t, \vec{r}, \vec{p})$  is given by

$$f^{(1)}(t, \vec{r}, \vec{p}) = -\tau_R f^{(0)}(t, \vec{r}, \vec{p}) \left[ \frac{1}{\rho(t, \vec{r})} \mathcal{D}\rho(t, \vec{r}) + \left( \frac{m\vec{u}_{\vec{p}}^2}{2k_B T(t, \vec{r})} - \frac{3}{2} \right) \frac{1}{T(t, \vec{r})} \mathcal{D}T(t, \vec{r}) \right. \\ \left. + \frac{m\vec{u}_{\vec{p}}^i}{k_B T(t, \vec{r})} \mathcal{D}\vec{v}^i(t, \vec{r}) - \frac{\vec{u}_{\vec{p}}}{k_B T(t, \vec{r})} \vec{F}(t, \vec{r}) \right]$$

where  $\mathcal{D} = \frac{\partial}{\partial t} + \frac{\vec{p}}{m} \vec{\nabla}_{\vec{r}}$  and we denote  $\vec{u}_{\vec{p}} = \vec{p}/m - \vec{v}(t, \vec{r})$ .

- ii) Exploit the equations of motions of ideal hydrodynamics

$$\left( \frac{\partial}{\partial t} + \vec{v}(t, \vec{r}) \vec{\nabla} \right) \rho(t, \vec{r}) = -\rho(t, \vec{r}) \vec{\nabla} \vec{v}(t, \vec{r}), \\ \frac{\partial}{\partial t} \vec{v}^i(t, \vec{r}) + \vec{v}^j(t, \vec{r}) \frac{\partial}{\partial r_j} \vec{v}^i(t, \vec{r}) = -\frac{1}{\rho(t, \vec{r})} \frac{\partial}{\partial r_i} P(t, \vec{r}) + \frac{1}{m} \vec{F}^i(t, \vec{r}), \\ \frac{\partial}{\partial t} e(t, \vec{r}) + \vec{\nabla} \cdot (e(t, \vec{r}) \vec{v}(t, \vec{r})) = -P(t, \vec{r}) \vec{\nabla} \vec{v}(t, \vec{r}),$$

to show that the expression for  $f^{(1)}(t, \vec{r}, \vec{p})$  can be compactly expressed as

$$f^{(1)}(t, \vec{r}, \vec{p}) = -\tau_R f^{(0)}(t, \vec{r}, \vec{p}) \left[ \left( \frac{m\vec{u}_{\vec{p}}^2}{2k_B T(t, \vec{r})} - \frac{5}{2} \right) \frac{\vec{u}_{\vec{p}} \vec{\nabla} T(t, \vec{r})}{T(t, \vec{r})} \right. \\ \left. + \frac{m}{2k_B T(t, \vec{r})} \left( \frac{\partial v_i}{\partial r_j}(t, \vec{r}) + \frac{\partial v_j}{\partial r_i}(t, \vec{r}) \right) \left( \vec{u}_{\vec{p}}^i \vec{u}_{\vec{p}}^j - \frac{1}{3} \delta^{ij} \vec{u}_{\vec{p}}^2 \right) \right]$$

(Hint: Express the energy density and pressure in terms of the temperatures and densities, by using the equations of state  $e(t, \vec{r}) = \frac{3}{2} n(t, \vec{r}) k_B T(t, \vec{r})$  and  $P(t, \vec{r}) = n(t, \vec{r}) k_B T(t, \vec{r})$ ) to derive the evolution equation for the temperature  $T(t, \vec{r})$

$$\left( \frac{\partial}{\partial t} + \vec{v}(t, \vec{r}) \vec{\nabla} \right) T(t, \vec{r}) = -\frac{2}{3} T(t, \vec{r}) \vec{\nabla} \vec{v}(t, \vec{r}), \quad (2)$$

Eliminate all time derivatives in the expression for  $f^{(1)}$  in favor of spatial derivatives using the ideal equations of motion. )

- iii) Verify explicitly that  $f^{(1)}(t, \vec{r}, \vec{p})$  does not contribute to the quantities  $\rho(t, \vec{r})$ ,  $v(t, \vec{r})$  and  $e(t, \vec{r})$ . (Hint: Change integration variables to  $d^3\vec{u}_{\vec{p}}$  and consider the symmetries of the integrand)

- iv) Calculate the heat conductivity  $\kappa$  in the constitutive relation  $\vec{J}_U = -\kappa \vec{\nabla} T(t, \vec{r})$

$$\left( \text{Hint: } \int_{\vec{p}} \frac{(m\vec{u}_{\vec{p}})^2}{2m} (m\vec{u}_{\vec{p}})^2 \left( \frac{(m\vec{u}_{\vec{p}})^2}{2mk_B T} - \frac{5}{2} \right) f^{(0)} = \frac{15}{2} mn (k_B T)^2 \right)$$