Non-equilibrium physics WS 18/19 – Exercise Sheet 6:

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1 Discussion:

i) What is a *mean-field approximation* and under what conditions is it applicable to a systems of interacting particles? What is the physical meaning of the different terms in the Boltzmann equation? What are the assumptions underlying the derivation of the Boltzmann equations?

2 In-class problems:

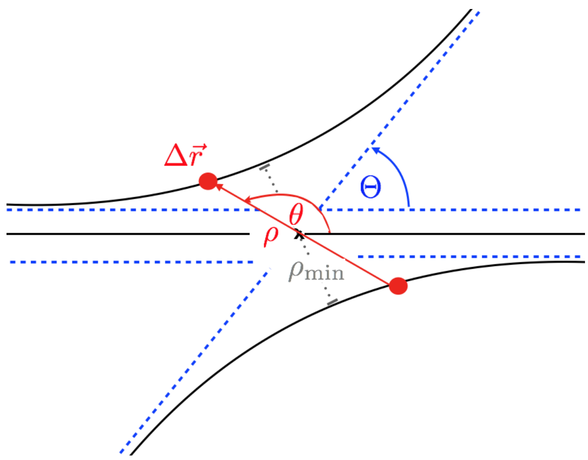
2.1 Classical cross-section calculation

We consider the interaction of two classical particles, in the presence of a radially symmetric two-particle interaction potential $W(|\vec{r}_1 - \vec{r}_2|)$.

i) Denoting the positions and momenta of the particles as $\vec{r}_{1/2} = \vec{R} \pm \frac{1}{2}\Delta\vec{r}$, show that the total energy can be separated into a center of mass motion and the energy of the two-particle system in the center-of-mass frame according to

$$E = \frac{1}{2}M\dot{\vec{R}}^2 + E_{\rm com} \ , \qquad E_{\rm com} = \frac{1}{2}\mu\Delta\dot{\vec{r}}^2 + W(|\Delta\vec{r}|) \ , \qquad M = 2m \ , \qquad \mu = \frac{m}{2} \ .$$

Since the center of mass motion decouples, the evolution of the two-particle system in the center-of-mass frame effectively becomes a one-body problem, described by the relative coordinate $\Delta\vec{r}$ and relative velocity $\Delta\dot{\vec{r}}$. Conventionally, one uses spherical coordinates (ρ, θ, ϕ) to describe the two body dynamics. Based on the azimuthal symmetry of the problem we can set $\phi = 0$ without loss of generality and study the dynamics in the $x - z$ plane as illustrated in the figure. Since the angular momentum of the two-body system is conserved in the interaction via a central force, we have two conserved quantities the angular momentum $\vec{L}$ and energy $E_{\rm com}$ in the center of mass frame, given by



$$\vec{L} = \mu\left(\Delta\vec{r} \times \Delta\dot{\vec{r}}\right) \qquad E_{\rm com} = \frac{1}{2}\mu\Delta\dot{\vec{r}}^2 + W(|\Delta\vec{r}|) \ .$$

ii) Write down the explicit expressions for $\vec{L}$ and E_{com} in terms of ρ, θ and $\dot{\rho}, \dot{\theta}$. By parametrizing the one-dimensional trajectory in term of the angle θ, i.e. expressing $\rho = \rho(\theta)$, show that the center of mass energy can be expressed as

$$E_{\text{com}}(\rho, \theta) = \frac{\vec{L}^2}{2\mu\rho^2}\left[\frac{1}{\rho^2}\left(\frac{\partial\rho}{\partial\theta}\right)^2 + 1\right] + W(\rho)\ .$$

iii) Since the angular momentum and center of mass energy are conserved, one has $E_{\text{com}}(\rho, \theta) = E_{\text{com}}$ and $\vec{L}^2 = 2b^2\mu E_{\text{com}}$, where $b = |\vec{b}|$ denotes the impact parameter. Exploiting this behavior and changing variables to $u(\rho) = 1/\rho$, show that the scattering angle Θ can be determined as

$$\Theta(b) = \pi - 2\int_0^{u_{\max}} du\ \frac{b}{\sqrt{1 - b^2u^2 - W(1/u)/E_{\text{com}}}}\ .$$

What is the meaning of $u_{\max}$ in this expression?

iv) Determine the scattering angle $\Theta(b)$ for the scattering of two inpenetrable hard spheres, described by the potential

$$W(\rho) = \begin{cases} 0 & \rho > 2R \\ \infty & \rho < 2R \end{cases}$$

$\left(\text{Hint: What is } u_{\max} \text{ in this case? One has } \int_0^a \frac{1}{\sqrt{1-x^2}} = arcsin(a) \text{ for } |a| < 1\right)$

v) Determine the differential cross-section $\frac{d\sigma}{d\Omega}(E_{\text{com}}, \phi, \Theta)$ relating the incoming and outgoing flux according to $\int d^2\vec{b} = \int d\phi \int b\ db = \int d\Omega\ \frac{d\sigma}{d\Omega}$, where $d\Omega = d\phi\ d\Theta \sin(\Theta)$.

3 Homework problems:

3.1 Free-streaming

Consider a *free-streaming* system of particles, which is described by the collisionless Boltzmann equation in the absence of external forces.

i) Show that the general solution for the single particle distribution of a free-streaming system is of the form $f(t,\vec{r},\vec{p}) = f\left(t_0, \vec{r} - \frac{\vec{p}}{m}(t-t_0), \vec{p}\right)$

3.2 AC conductivity of a collisionless plasma

We consider a plasma of a single species of charged particles, described by the collisionless Boltzmann equation in position ($\vec{r}$) and velocity ($\vec{v} = \frac{\vec{p}_{\rm kin}}{m}$) space

$$\left[\frac{\partial}{\partial t} + \vec{v}\cdot\vec{\nabla}_{\vec{r}} + \frac{e}{m}\left(\vec{E}(t,\vec{r}) + \frac{\vec{v}}{c}\times\vec{B}(t,\vec{r})\right)\vec{\nabla}_{\vec{v}}\right] f(t,\vec{r},\vec{v}) = 0\ , \tag{1}$$

in the presence of a small external electric field

$$\vec{E}(t,\vec{r}) = \lambda \vec{E}_0 e^{i[\vec{k}\vec{r}-\omega t]}\ .$$

and no magnetic field $\vec{B}(t,\vec{r})$.

i) Develop a microscopic picture of the motion of the charged particles over a time scale $\Delta t_\omega = \frac{2\pi}{\omega}$. Discuss how the collisionless approximation can be justified in the limit $v_{\rm th}\Delta t_\omega \ll l_{\rm mfp}$ and why the approximation is not suitable to compute the response to a static electric field ($\omega = 0$).

Since the general solution to this problem is hard to find, we will instead construct the solution perturbatively, by expanding the single particle distribution according to

$$f(t,\vec{r},\vec{v}) = f_0(\vec{v}) + \lambda \delta f_{\vec{k},\omega}(t,\vec{r},\vec{v})$$

where $\delta f_{\vec{k},\omega}(t,\vec{r},\vec{v}) \ll f_0(\vec{v})$ is a small perturbation generated in response to the external electric field. We consider as our expansion point, an equilibrium distribution of the form

$$f_0(\vec{v}) = n_0 e^{\frac{-m\vec{v}^2}{2k_BT}} \left(\frac{2\pi\hbar^2}{mk_BT}\right)^{3/2}\ ,$$

where n_0 denotes the local particle number density.

ii) Show that the background distribution f_0 satisfies the evolution equation $\left[\frac{\partial}{\partial t} + \vec{v}\cdot\vec{\nabla}_{\vec{r}}\right] f_0 = 0$. Construct the evolution equation for the perturbations $\delta f_{\vec{k},\omega}(t,\vec{r},\vec{v})$ by collecting all residual terms of $O(\lambda)$ in Eq. (1), neglecting terms of order $O(\lambda^2)$ and higher.

Since in the long time limit, the space-time dependence of the perturbation is expected to follow that of the external electric field, we will search for solutions of the form

$$\delta f_{\vec{k},\omega}(t,\vec{r},\vec{v}) = \delta f_{\vec{k},\omega}(\vec{v}) e^{i[\vec{k}\vec{r}-(\omega+i\epsilon)t]}\ ,$$

where in the above expression $\epsilon > 0$ is inserted to ensure that $\delta f_{\vec{k},\omega}(\vec{v}) \to 0$ in the limit $t \to -\infty$, and we will take the limit $\epsilon \to 0$ in the final step of our calculation.

iii) Show that the solution for $\delta f_{\vec{k},\omega}(\vec{v})$ can be expressed as

$$\delta f_{\vec{k},\omega}(\vec{v}) = \frac{ie}{k_BT}\frac{\vec{E}_0\cdot\vec{v}}{\omega + i\epsilon - \vec{v}\cdot\vec{k}} f_0(\vec{v}) \tag{2}$$

Based on the solution in Eq. (2) we will now proceed to calculate the conductivity tensor $\sigma(\vec{k},\omega)$, which according to $J^i = \sigma^{ij} E^j$ relates the induced current

$$J^i = e \int \frac{m^3 d^3\vec{v}}{(2\pi\hbar)^3}\ v^i\ f(t,\vec{r},\vec{v})$$

to the external electric field $\vec{E}$.

iv) Show the components of the conductivity can be expressed as

$$\sigma^{ij} = \frac{ie^2m^3}{k_BT} \int \frac{d^3\vec{v}}{(2\pi\hbar)^3} \frac{\vec{v}^i\ \vec{v}^j}{\omega + i\epsilon - \vec{v}\cdot\vec{k}}\ f_0(\vec{v})$$

v) Show that the conductivity tensor σ is of the diagonal form $\sigma = diag(\sigma_T, \sigma_T, \sigma_L)$ where $\sigma_{T/L}$ denote the transverse (T) and longitudinal components (L) w.r.t. the wave-vector $\vec{k}$.

vi) Calculate the real and imaginary part of the transverse conductivity $\sigma_T(\vec{k},\omega)$

vii) Calculate the real and imaginary part of the longitudinal conductivity $\sigma_L(\vec{k},\omega)$

Some hints on the evaluation of integrals

a) Separate the integral $\int d^3\vec{v}$ into integrations over the longitudinal $v_z = \frac{\vec{k}\cdot\vec{v}}{|\vec{k}|}$ and transverse velocities and first perform the integration over the transverse velocities.

b) Express the integral over the longitudinal velocity v_z in the form

$$\lim_{\epsilon\to 0} \int_{-\infty}^{\infty} dx\ \frac{f(x)}{x+i\epsilon}$$

which can be evaluated as

$$\lim_{\epsilon\to 0} \int_{-\infty}^{\infty} dx\ \frac{f(x)}{x+i\epsilon} = -i\pi f(0) + \mathrm{P} \int_{-\infty}^{\infty} dx\ \frac{f(x)}{x}$$

where the principle value (P) of the integral can be calculated according to

$$\mathrm{P} \int_{-\infty}^{\infty} dx\ \frac{f(x)}{x} = \int_0^{\infty} dx\ \frac{f(x)-f(-x)}{x}\ .$$

c) Some useful formulae for the remaining integrals include ($\sigma > 0$)

$$\begin{aligned}
\frac{1}{\sqrt{2\pi\sigma^2}} \int_0^{\infty} dx\ \left(\frac{x}{\sigma}\right)\ \sinh(ax/\sigma^2)\ e^{-\frac{x^2}{2\sigma^2}} &= \frac{a}{2\sigma} e^{\frac{a^2}{2\sigma^2}}\ , \\
\frac{1}{\sqrt{2\pi\sigma^2}} \int_0^{\infty} dx\ \frac{\sinh(ax/\sigma^2)}{x/\sigma}\ e^{-\frac{x^2}{2\sigma^2}} &= \frac{1}{2}\sqrt{\frac{\pi}{2}} Erfi\left(\frac{a}{\sqrt{2}\sigma}\right)\ , \\
\frac{1}{\sqrt{2\pi\sigma^2}} \int_0^{\infty} dx\ \cosh(ax/\sigma^2)\ e^{-\frac{x^2}{2\sigma^2}} &= \frac{1}{2} e^{\frac{a^2}{2\sigma^2}}\ ,
\end{aligned} \tag{3}$$