# Non-equilibrium physics WS 18/19 – Exercise Sheet 4:

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## 1 Discussion:

i) How is a microscopic state characterized in classical and quantum physics? What is the significance of the *phase-space distribution* and the *density operator*?

## 2 In-class problems:

### 2.1 Evolution of phase-space density

We first consider a statistical ensemble of free particles, with Hamiltonian $H(\{x_i\},\{p_i\}) = \sum_i \frac{p_i^2}{2m}$ and initial phase-space distribution $f(t_0,\{x_i\},\{p_i\}) = f_0(\{x_i\},\{p_i\})$ at initial time $t_0 = 0$.

i) Write down the Liouville equation for the time evolution of the phase-space distribution and construct the formal solution in terms of the Liouville operator.

ii) Calculate the the action of the Liouville operator $\mathcal{L}f$ and $\mathcal{L}^2 f$ on the phase-space distribution $f$. Deduce from this result the form $\mathcal{L}^n f$.

iii) Construct the explicit solution to the solution Liouville equation.
$\Big($Hint: $e^{a\partial/\partial_x} f(x) = \sum_{n=0}^{\infty} \frac{a^n}{n!} f^{(n)}(x) = f(x+a)\Big)$

Next consider a statistical ensemble of classical harmonic oscillators, with Hamiltonian $H(x,p) = \frac{p^2}{2m} + \frac{1}{2}m\omega^2 x^2$ and initial phase-space distribution $f(t_0,x,p) = f_0(x,p)$ at initial time $t_0 = 0$.

iv) Write down Hamilton's equations of motion for $\dot{x}, \dot{p}$ and solve them for general initial conditions $x(0) = x_0$ and $p(0) = p_0$.

v) Exploit your result in iv) to compute the time evolution of the phase-space distribution $f(t,x,p)$

# 3 Homework problems:

## 3.1 Entropy production in non-relativistic hydrodynamics

i) Show that the entropy production rate $\sigma_S = \sum_i (J_i - J_i^{eq}) F_i$ for a Newtonian fluid is given by

$$\sigma_S = \kappa T^2 \left( \vec{\nabla} \frac{1}{T} \right)^2 + \frac{\zeta}{T} (\vec{\nabla} \vec{v})^2 + \frac{2\eta}{T} \sum_{\alpha\beta} (\sigma^{\alpha\beta})^2$$

ii) Determine the entropy current $J_S$ in the local rest-frame, according to the general relation

$$J_S|_{LRF} = \sum_i Y_i J_i \ .$$

$\Big($Hint: Which intensity quantities $Y_i$ and fluxes $J_i$ are non-zero in the LRF? $\Big)$

iii) Since the entropy density is frame independent, the entropy current in an arbitrary frame is given by

$$\vec{J}_S = \vec{J}_S \Big|_{LRF} + s\vec{v} \ .$$

Based on this result, along with the results of i) and ii), write down the explicit form of the entropy balance equation for a Newtonian fluid.

## 3.2 Evolution of expectation values in classical & quantum theories

Consider a classical system described by a Hamiltonian $H(x,p) = \frac{p^2}{2m} + V(x)$ and its quantum mechanical analogue, where the phase-space variables $x, p$ are replaced by operators $\hat{x}$ and $\hat{p}$. Note that quantum mechanically $[\hat{x}, \hat{p}] = i\hbar$ which implies the useful identity $[\hat{p}, V(\hat{x})] = -i\hbar V'(\hat{x})$ that you can use without proof.

i) Determine the classical and quantum evolution equations for the observables

$$\left\langle \frac{1}{2} x^2 \right\rangle, \qquad \left\langle \frac{1}{2} p^2 \right\rangle \quad and \quad \langle xp + px \rangle \ .$$

$\Big($Hints: Commutators can be simplifed according to $[AB, C] = A[B, C] + [A, C]B$. $\Big)$

ii) Specialize now on the case of a harmonic oscillator $V(x) = \frac{1}{2} m\omega^2 x^2$ and solve the coupled set of evolution equations for $\langle E_{pot} \rangle = \langle \frac{1}{2} m\omega^2 x^2 \rangle$, $\langle E_{kin} \rangle = \langle \frac{p^2}{2m} \rangle$ and $\langle xp + px \rangle$. How do the results compare between the classical and quantum theory?
$\Big($Hint: Energy conservation $\Big)$