Non-equilibrium physics WS 18/19 – Exercise Sheet 11:

Universität Bielefeld
Instructors: Jun.-Prof. Dr. S. Schlichting, D. Schröder

1 Discussion:

i) What are the physical differences between stationary turbulent solutions and equilibrium solutions of kinetic equations? What are direct and inverse cascades?

2 In-class problems:

2.1 Kolmogorov-Zakharov Spectra

Consider the three wave kinetic equation $\partial_t n(k,t) = I[n](k,t)$ for a statistically homogenous and isotropic system of waves, with power law dispersion $\omega(k) = \omega_0(k/k_0)^z$ with the collision integral given by

$$I[n](k,t) = \frac{1}{2}\int d^dk_1 \int d^dk_2 \Big\{ \tilde{w}(k \to k_1k_2)\Big[n(k_1,t)n(k_2,t) - n(k,t)\big(n(k_1,t)+n(k_2,t)\big)\Big] - \tilde{w}(k_1 \to k_2k)\Big[n(k_2,t)n(k,t) - n(k_1,t)\big(n(k_2,t)+n(k,t)\big)\Big] - \tilde{w}(k_2 \to kk_1)\Big[n(k,t)n(k_1,t) - n(k_2,t)\big(n(k,t)+n(k_1,t)\big)\Big]\Big\} \quad (1)$$

where $\widetilde{w}(k \to k_1k_2) = (2\pi)\delta(\omega(k)-\omega(k_1)-\omega(k_2))\delta^{(d)}(k-k_1-k_2)|V(k,k_1,k_2)|^2$ for a scale invariant matrix element $V(\lambda k, \lambda k_1, \lambda k_2) = \lambda^m V(k,k_1,k_2)$ with the usual symmetry properties $V(k,k_1,k_2) = V(k,k_2,k_1)$.

i) Show that for a spectrum of the Kolmogorov-Zakharov (KZ) form $n(k,t) = n_{\rm KZ}(k) = n_0\left(\frac{k}{k_0}\right)^{-s_0}$ the energy flux $J_k(\Lambda,t) = -\int_0^\Lambda d^dk\ \omega(k)\ I[n](k,t)$ through a momentum shell can be expressed as

$$J_k(\Lambda) = -\omega_0 k_0^{d-(\xi+z)}\Omega^{(d)} \int_0^\Lambda dk\ k^{\xi+z-1} I[n_{\rm KZ}](k_0)\ ,$$

with $\xi = 2d - z + 2m - 2s_0$.

ii) Determine the scaling exponent s_0 of the KZ spectrum based on the condition that the energy flux $J_k(\Lambda,t) = -\int_0^\Lambda d^dk\ \omega(k)\ I[n](k,t)$ through a momentum shell becomes scale (Λ) invariant. Explain why this condition is necessary for $n(k,t) = n_{\rm KZ}(k)$ to be a stationary turbulent solution.

3 Homework problems:

3.1 Self-similar solutions

Consider the decaying turbulence of a statistically homogenous and isotropic system of waves described by the three wave kinetic equation $\partial_t n(k,t) = I[n](k,t)$.

iii) Determine the dynamical scaling exponents α, β of the self-similar solutions

$$n(k,t) = (t/t_0)^{\alpha} n_S\left(k(t/t_0)^{\beta}\right) ,$$

of the kinetic equation when a) no energy is injected or removed from the system and b) energy is injected into the system at a constant rate $\dot{\epsilon} = \dot{\epsilon}_0$.

3.2 Stationary turbulence of capillary waves on shallow water

Consider again the three wave kinetic equation $\partial_t n(k,t) = I[n](k,t)$ for a statistically homogenous and isotropic system of waves.

i) Show that for a spectrum of the Kolmogorov-Zakharov (KZ) form $n(k,t) = n_{\text{KZ}}(k) = n_0\left(\frac{k}{k_0}\right)^{-s_0}$ the collision kernel $I[n](k,t)$ in Eq. (1) evaluated at $k = k_0$ can be expressed as

$$I[n_{\text{KZ}}](k_0) = \frac{n_0^2}{2\Omega^{(d)}} \int d\Omega_0 \int d^d k_1 \int d^d k_2 \ \tilde{w}(k_0 \to k_1 k_2)$$
$$\left[\left|\frac{k_0}{k_1}\right|^{s_0}\left|\frac{k_0}{k_2}\right|^{s_0} - \left(\left|\frac{k_0}{k_1}\right|^{s_0} + \left|\frac{k_0}{k_2}\right|^{s_0}\right)\right] \times \left[1 - \left|\frac{k_0}{k_1}\right|^{\xi} - \left|\frac{k_0}{k_2}\right|^{\xi}\right]$$

with $\xi = 2d - z + 2m - 2s_0$. (Hint: Zakharov transformation)

ii) Show that in the limit $s_0 \to m + d$ the energy flux $J_k(\Lambda, t) = -\int_0^{\Lambda} d^d k \ \omega(k) \ I[n](k,t)$ through a momentum shell is given by

$$J_k(\Lambda) = \frac{\omega_0 n_0^2 k_0^d}{2} \int d\Omega_0 \int d^d k_1 \int d^d k_2 \ \tilde{w}(k_0 \to k_1 k_2) \tag{2}$$
$$\times \left[\left|\frac{k_0}{k_1}\right|^{s_0}\left|\frac{k_0}{k_2}\right|^{s_0} - \left(\left|\frac{k_0}{k_1}\right|^{s_0} + \left|\frac{k_0}{k_2}\right|^{s_0}\right)\right]\left[\left|\frac{k_0}{k_1}\right|^{-z}\log\left|\frac{k_0}{k_1}\right| + \left|\frac{k_0}{k_2}\right|^{-z}\log\left|\frac{k_0}{k_2}\right|\right]$$

Now consider as a particular example the dynamics of surface waves in shallow water, as described by a three wave kinetic equation in $d = 2$ dimensions. In this particular case, the dispersion relation is given by $\omega(k) = \omega_0 k^2/k_0^2$ and the interaction matrix element takes a particularly simple form $V(k, k_1, k_2) = V_0 k^2/k_0^2$, with $\omega_0/k_0^2 = \left(\frac{\sigma h}{\rho}\right)^{1/2}$ and $V_0/k_0^2 = \frac{1}{8\pi}\left(\frac{\sigma}{4\rho h}\right)^{1/4}$ where ρ and σ denote the density and surface tension of the fluid, and h is the height of the container.

iii) Determine the scaling exponent s_0 of the KZ spectrum for stationary turbulence of capillary waves on shallow water (Check: $s_0 = 4$)

iv) Show that for two-dimensional dynamics of capillary waves on shallow water, the phase-space integrations in Eq. (2) can be performed according to

$$\int d\Omega_0 \int d^2 k_1 \int d^2 k_2 \ \tilde{w}(k_0 \to k_1 k_2) f\left(\left(\frac{k_1}{k_0}\right), \left(\frac{k_2}{k_0}\right)\right) = \frac{(2\pi)^2 k_0^2 V_0^2}{2\omega_0} \int_0^1 dx \frac{f\left(x, \sqrt{1-x^2}\right)}{\sqrt{1-x^2}} ,$$

v) Calculate the energy flux through a momentum shell $J_k(\Lambda)$ in Eq. (2) for stationary turbulence of capillary waves on shallow water (Check: $J_k(\Lambda) = 2\pi^3 n_0^2 k_0^4 V_0^2$)

vi) Show that if energy is injected into the system at a constant rate P, the stationary turbulent solution in the intertial range of the cascade takes the form

$$n_{KZ}(k) = \frac{8P^{1/2}}{\sqrt{\pi}} \left(\frac{\rho h}{\sigma}\right)^{1/4} k^{-4} \,, \tag{3}$$

3.3 Formulae:

$$x^{a+\epsilon} \simeq x^a + \epsilon x^a \log(x) + \mathcal{O}(\epsilon^2) \,,$$

$$\int_0^1 dx \frac{1}{\sqrt{1-x^2}} \left[x^{-4}\sqrt{(1-x^2)}^{-4} - \left(x^{-4} + \sqrt{1-x^2}^{-4} \right) \right] \left[x^2 \log\left(\frac{1}{x}\right) + (1-x^2)\log\left(\frac{1}{\sqrt{1-x^2}}\right) \right] = 2\pi$$