Recap:

Discussed dynamics of Newtonian fluids

Described by conservation laws for ρ, $\rho\vec{v}$, $\varepsilon + \frac{1}{2}\rho\vec{v}^2$

$$\frac{\partial \rho}{\partial t} + \vec{\nabla} \vec{J}_\rho = 0$$

$$\frac{\partial (\rho\vec{v})}{\partial t} + \vec{\nabla} \underline{\underline{J}}_{\vec{p}} = 0$$

$$\frac{\partial}{\partial t}\left(\varepsilon + \frac{1}{2}\rho\vec{v}^2\right) + \vec{\nabla}\vec{J}_E = 0$$

closed by constitutive relations for fluxes

$$\vec{J}_\rho = \rho\vec{v}$$ (no non-equilibrium corrections due to definition of $\vec{v}$)

$$J_p^{\alpha\beta} = \Pi^{\alpha\beta} + \rho v^\alpha v^\beta$$

stress tensor

$$\underline{\underline{\Pi}} = \left(P - \zeta(\vec{\nabla}\cdot\vec{v})\right)\underline{\underline{1}} - 2\eta\,\underline{\underline{\sigma}}$$

with $\sigma^{\alpha\beta} = \frac{1}{2}\left(\frac{\partial v^\alpha}{\partial x^\beta} + \frac{\partial v^\beta}{\partial x^\alpha}\right) - \frac{1}{3}(\vec{\nabla}\cdot\vec{v})\,\delta^{\alpha\beta}$

$$\vec{J}_E = \left(e + \frac{1}{2}\rho\vec{v}^2\right)\vec{v} + \underline{\underline{\Pi}}\,\vec{v} - \kappa\vec{\nabla}T$$

We found

$$\frac{\partial \rho}{\partial t} + \vec{\nabla}(\rho \vec{v}) = 0 \qquad \text{continuity equation}$$

and assuming ζ, η to be constant

$$\boxed{\rho\left[\frac{\partial \vec{v}}{\partial t} + (\vec{v}\cdot\vec{\nabla})\vec{v}\right] = -\vec{\nabla} P + \zeta\, \vec{\nabla}(\vec{\nabla}\cdot\vec{v}) + \eta\underbrace{\left(\Delta\vec{v} + \frac{1}{3}\vec{\nabla}(\vec{\nabla}\cdot\vec{v})\right)}_{\text{conservative + friction forces}}}$$

<u>Navier - Stokes equation</u>

In particular for an <u>incompressible fluid</u>

$$\rho = \text{const} \underset{\text{cont eqn}}{\Longrightarrow} \boxed{\vec{\nabla}\vec{v} = 0}$$

$$\rho\left[\frac{\partial \vec{v}}{\partial t} + (\vec{v}\cdot\vec{\nabla})\vec{v}\right] = -\vec{\nabla} P + \eta\, \Delta\vec{v}$$

Similarly we can write down the energy balance equation

$$\frac{\partial}{\partial t}\left(\varepsilon + \frac{1}{2}\rho \vec{v}^2\right) + \vec{\nabla}\vec{J}_E = 0$$

$$\vec{J}_E = \left(e + \frac{1}{2}\rho\vec{v}^2\right)\vec{v} + \underline{\Pi}\,\vec{v} + \vec{J}_q$$

with $\underline{\underline{\Pi}} = \left(P - \zeta\,(\vec{\nabla}\cdot v)\right)\underline{\mathbb{1}} - 2\eta\,\underline{\sigma}$

$$\vec{J}_q = -\varkappa\,\vec{\nabla}T$$

So using

$$2\eta\,\sigma^{\alpha\beta}v^{\beta} = \left[\eta\left(\frac{\partial v_\alpha}{\partial x^\beta} + \frac{\partial v^\beta}{\partial x^\alpha}\right) - \frac{2}{3}\eta\,(\vec{\nabla}\cdot\vec{v})\,\delta^{\alpha\beta}\right]v^\beta$$

$$= \eta\,(\vec{v}\cdot\vec{\nabla})\vec{v}^{\alpha} + \eta\,\vec{\nabla}_\alpha\left(\frac{1}{2}\vec{v}^2\right) - \frac{2}{3}\eta\,(\vec{\nabla}\vec{v})\,v^\alpha$$

we get

$$\frac{\partial}{\partial t}\left(\varepsilon + \frac{1}{2}\rho\vec{v}^2\right) + \vec{\nabla}\Big[\left(e + P + \frac{1}{2}\rho\vec{v}^2\right)\vec{v}$$

$$-\,\eta\left((\vec{v}\cdot\vec{\nabla})\vec{v} + \vec{\nabla}\left(\frac{1}{2}\vec{v}^2\right) - \frac{2}{3}\vec{v}(\vec{\nabla}\cdot\vec{v})\right)$$

$$-\,\zeta\vec{v}(\vec{\nabla}\cdot\vec{v}) - \varkappa\,\vec{\nabla}T\Big] = 0$$

Entropy production in NS eqn

Since entropy is frame independent
can look at entropy production in
LRF which simplifies the calculation

$$\cancel{\sigma_s = \sum_i J_i F_i}$$

$$\sigma_s = \sum_i \left(J_i - J_i^{eq}\right) F_i$$

$$= \vec{J}_u \vec{\nabla}\left(\frac{1}{T}\right) + J_{\vec{p}}^{\alpha\beta}\Big|_{NEQ} \left(-\frac{1}{T}\frac{\partial v_\alpha}{\partial x_\beta}\right)$$

$$= -\kappa\left(\vec{\nabla}T\right)\vec{\nabla}\left(\frac{1}{T}\right) + \left(\pi^{\alpha\beta} - P\delta^{\alpha\beta}\right)\left(-\frac{1}{T}\frac{\partial v_\alpha}{\partial x_\beta}\right)$$

$$= \kappa T^2\left(\vec{\nabla}\left(\frac{1}{T}\right)\right)^2 - \frac{1}{T}\left(\frac{\partial v_\alpha}{\partial x_\beta}\right)\left(-2\eta\sigma^{\alpha\beta} - \zeta\left(\vec{\nabla}\cdot\vec{v}\right)\delta^{\alpha\beta}\right)$$

$$= \kappa T^2\left(\vec{\nabla}\left(\frac{1}{T}\right)\right)^2 + \frac{\zeta}{T}\left(\vec{\nabla}\cdot\vec{v}\right)^2 + \frac{2\eta}{T}\sum_{\alpha\beta}\left(\sigma^{\alpha\beta}\right)^2$$

$$\geq 0 \quad \text{for } \kappa, \zeta, \eta \geq 0$$

Where

$$2\eta\,\sigma^{\alpha\beta}\frac{\partial v^\alpha}{\partial x^\beta} = 2\eta\,\sigma^{\alpha\beta}\Bigg(\underbrace{\frac{1}{2}\left(\frac{\partial v^\alpha}{\partial x^\beta} + \frac{\partial v^\beta}{\partial x^\alpha}\right)}_{\text{can add due to } \sigma^{\alpha\beta} = \sigma^{\beta\alpha}} - \underbrace{\frac{1}{3}\left(\vec{\nabla}\vec{v}\right)\delta^{\alpha\beta}}_{\text{can add due to } \delta^{\alpha\beta}\sigma^{\alpha\beta} = tr[\sigma] = 0}\Bigg)$$

Can now look at dissipation of kinetic energy in incompressible fluid

$$\frac{\partial}{\partial t} E_{kin} = \rho \int d^3r \; V_\alpha \left(\frac{\partial}{\partial t} V_\alpha\right)$$

$$\stackrel{NS\,eqn}{=} \int d^3r \; V_\alpha \left(-\rho \left(V_\beta \frac{\partial}{\partial x_\beta}\right) V^\alpha - \frac{\partial}{\partial x_\alpha} P + \eta \frac{\partial^2}{\partial x_\beta \partial x_\beta} V^\alpha \right)$$

now can rewrite

A) $$V_\alpha \frac{\partial}{\partial x_\alpha} P = \frac{\partial}{\partial x_\alpha} \left(V_\alpha P\right) - P \left(\frac{\partial}{\partial x_\alpha} V_\alpha\right)$$

B) $$V_\alpha \rho V_\beta \frac{\partial}{\partial x_\beta} V^\alpha = \rho V_\beta \frac{\partial}{\partial x_\beta}\left(\frac{1}{2}\vec{V}^2\right)$$

$$= \frac{\partial}{\partial x_\beta}\left(V_\beta \frac{1}{2}\rho \vec{V}^2\right)$$

$$- \frac{1}{2}\rho \vec{V}^2 \left(\frac{\partial}{\partial x_\beta} V_\beta\right)$$

C) $$V_\alpha \eta \frac{\partial^2}{\partial x_\beta \partial x_\beta} V^\alpha =$$

$$\eta \frac{\partial}{\partial x_\beta}\left(V_\alpha \frac{\partial}{\partial x_\beta} V^\alpha\right)$$

$$- \eta \left(\frac{\partial V_\alpha}{\partial x_\beta}\right)\left(\frac{\partial V_\alpha}{\partial x_\beta}\right)$$

note that we assumed $\eta = \text{const}$ in NS eqn

so

$$\frac{\partial}{\partial t} E_{kin} = \int d^3r \; \frac{\partial}{\partial x_\alpha}\left(-V_\alpha \left(P + \frac{1}{2}\rho \vec{V}^2\right) + \frac{1}{2}\eta \frac{\partial \vec{V}^2}{\partial x_\alpha}\right)$$

$$-\eta \int d^3r \left(\frac{\partial V_\alpha}{\partial x_\beta}\right)\left(\frac{\partial V_\alpha}{\partial x_\beta}\right)$$

Since the first term is a surface contribution
which vanishes if $v \to 0$ for $|\vec{r}| \to$ infinity
one is left with

$$\frac{\partial}{\partial t} E_{kin} = -\eta \int d^3r \left(\frac{\partial v_\alpha}{\partial x_\beta}\right)\left(\frac{\partial v_\alpha}{\partial x_\beta}\right) \leq 0$$

$\Rightarrow$ kinetic energy is dissipated into
internal energy of the fluid
due to viscous effects

Besides dynamical problems one important application of fluid dynamics is to study ~~stationary~~ states

Hydrostatic:

$$\frac{\partial}{\partial t} = 0 \quad , \quad \vec{v} = 0$$

Need to consider external forces as well

$$\vec{\nabla} P = \vec{F}_{ext}$$

as gravity $\vec{F}_{ext} = -g \rho \vec{e}_z$

$$\Rightarrow \quad \frac{\partial P}{\partial z} = -g \rho$$

ideal gas $\quad pV = N k_B T$

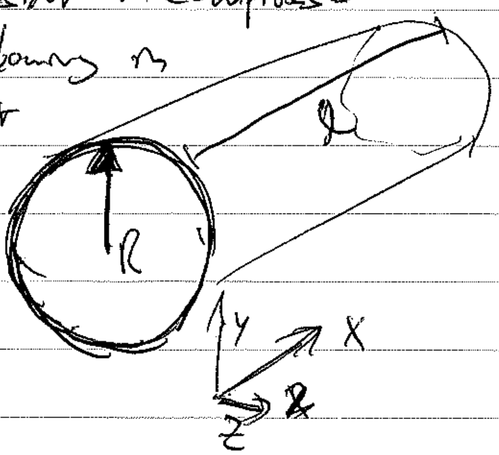
$$p = \frac{N}{V} k_B T = \rho \frac{k_B T}{m}$$

$$\Rightarrow \quad \frac{\partial}{\partial z} \rho = - \frac{mg}{k_B T} \rho$$

$$\rho = \rho_0 e^{-\frac{mg}{k_B T} z}$$

Stationary Flow in a pipe:

We consider incompressible fluid flowing in cylinder



$\vec{V} = V_x \vec{e}_x$

incompressibility

$$\frac{\partial \rho}{\partial t} + \vec{\nabla}(\rho \vec{V}) = 0$$

so for $\rho = \text{const}$ $\quad \vec{\nabla}\vec{V} = 0 \quad \Rightarrow \frac{\partial}{\partial x} V_x = 0$

$\vec{V} = V_x(r)\,\vec{e}_x$ where φ dependence vanishes by azimuthal symmetry

Now lets look at NS equation

$$\rho\left[\frac{\partial \vec{v}}{\partial t} + (\vec{V}\cdot\vec{\nabla})\vec{V}\right] = -\vec{\nabla}P + \zeta\,\vec{\nabla}(\vec{\nabla}\cdot\vec{V}) + \eta\left(\Delta\vec{V} + \frac{1}{3}\vec{\nabla}(\vec{\nabla}\vec{V})\right)$$

Since fluid is incompressible ($\vec{\nabla}\vec{V} = 0$) and flow is stationary ($\frac{\partial \vec{V}}{\partial t} = 0$)

$$\Rightarrow \quad \eta\,\Delta V_x = \frac{\partial P}{\partial x}$$

assuming const viscosity we get

$$\eta\,\Delta \underbrace{\frac{\partial}{\partial x} V_x}_{=0} = \frac{\partial^2 P}{\partial x^2} \quad \Rightarrow \quad \frac{\partial P}{\partial x} = \text{const}$$

So we conclude that

$$\frac{\partial p}{\partial x} = \frac{p_{out} - p_{in}}{l} = \frac{-\Delta p}{l}$$

where the pressure drops from in to out $\Delta p > 0$

$$\eta \Delta V_x = -\frac{\Delta p}{l}$$ where in cylinder coordinates $\Delta = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}\right)$

$$\Rightarrow \quad \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial V_x}{\partial r}\right) = -\frac{\Delta p}{\eta l}$$

So we obtain the general solution

$$r\frac{\partial V_x}{\partial r} = -\frac{1}{2}\frac{\Delta p}{\eta l} r^2 + a$$

$$V_x(r) = -\frac{1}{4}\frac{\Delta p}{\eta l} r^2 + a \log(r) + b$$

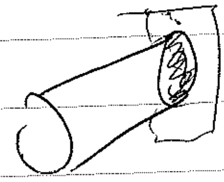
How to fix coefficients a, b?

1) Boundary conditions require $V_x(R) = 0$ at walls of cylinder

2) Flow ~~velocity~~ velocity should be finite in center $\lim_{r \to 0} \log(r) = -\infty \Rightarrow a = 0$

$$\boxed{V_x(r) = \frac{1}{4}\frac{\Delta p}{\eta l}\left(R^2 - r^2\right)}$$

Can also calculate mass flux through pipe



$$Q = \int d^2\vec{\sigma}\, \rho \vec{v}$$

"discharge"

$$= 2\pi \rho \int dr\, r\, v_x(r)$$

$$= \frac{\pi \Delta p}{8 \eta l} R^4 \rho$$

Bigger pipes transport more $(\propto R^4)$

a) because they are bigger

b) because the slow flow near the wall has a smaller impact on the stream in the ~~near~~ center