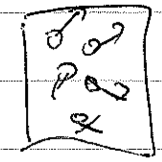
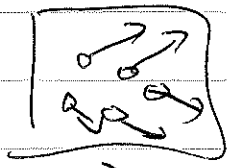
We started to discuss the dynamics of simple non-relativistic fluids

→ now independent systems can move



$\vec{P} = 0$

LRF



$\vec{P} \neq 0$

Lab frame

Since entropy is frame independent
needed to modify our notion of energy density
by dividing into internal energy & kinetic energy density

→ ~~[illegible]~~ intensive quantity

$$y_{\vec{P}} = -\frac{\vec{v}}{T}$$

Started to construct general form
of the constitutive relations: equilibrium fluxes

symmetries in LRF $\longrightarrow$ $\vec{J}^{eq}_{N}\big|_{LRF} = 0 \quad \vec{J}^{eq}_{E}\big|_{LRF} = 0$

$$J^{eq\,\alpha\beta}_{P}\big|_{LRF} = P\,\delta^{\alpha\beta}$$

In order to construct constitutive relations in arbitrary frame, we looked at behaviour under infinitesimal Galilei transformations $(d\vec{w})$

$$dJ_N^\alpha = n\, dw^\alpha$$

$$dJ_E^\alpha = J_{\vec{p}}^{\alpha\beta} dw^\beta + \left(\varepsilon + \tfrac{1}{2}\vec{w}^2\right) dw^\alpha$$

$$dJ_{\vec{p}}^{\alpha\beta} = m J_N^\beta dw^\alpha + \rho w^\alpha dw^\beta$$

which upon integrating from $\int_0^{\vec{v}} d\vec{w}$ i.e. from LRF to lab frame

$$J_N^\alpha = \underbrace{J_N^\alpha\big|_{LRF}}_{=0} + n\vec{v}$$

$$J_{\vec{p}}^{\alpha\beta} = J_{\vec{p}}^{\alpha\beta}\big|_{LRF} + \rho v^\alpha v^\beta$$

$$J_E^\alpha = J_E^\alpha\big|_{LRF} + J_p^{\alpha\beta}\big|_{LRF} v^\beta$$
$$+ \left(e + \tfrac{1}{2}\rho\vec{v}^2\right) v^\alpha$$

such that for equilibrium fluxes $(\underbrace{\vec{J}_N = \vec{J}_E = 0,\ J_{\vec{p}} = P\underline{\underline{1}}}_{LRF})$

$$J_N^\alpha\big|_{eq} = n\vec{v} \qquad J_{\vec{p}}^{\alpha\beta}\big|_{eq} = P\delta^{\alpha\beta} + \rho v^\alpha v^\beta$$

$$J_E^\alpha = \left(e + P + \tfrac{1}{2}\rho\vec{v}^2\right) v^\alpha$$

Now we should also consider possible out-of-equilibrium modifications in the linear (Markovian) transport regime

If we promote $\vec{v}$ to a fluid dynamical variable (i.e. consider $\vec{v}$ instead of the momentum) then we can define $\vec{v}$ such that $\vec{v} =$ average velocity of particles in fluid cell

$$\Rightarrow \quad \vec{v} = 0 \text{ in LRF}$$

and since particles are not moving on average in LRF

$$\vec{J}_N \Big|_{LRF} \equiv 0$$

No corrections to $\boxed{\vec{J}_N = n\vec{v}}$ in any frame

Likewise for momentum flux

$$\vec{J}_{\vec{p}}^{\alpha\beta}\Big|_{eq} \equiv \underbrace{P\delta^{\alpha\beta}} + \underbrace{\rho v^\alpha v^\beta}$$

can be modified due to non-eq.

does not receive corrections because purely from avg. motion of particles

$$P\delta^{\alpha\beta} \rightarrow \Pi^{\alpha\beta}$$

"stress tensor"

Similarly

$$\vec{J}_E = \underbrace{\left(e + \tfrac{1}{2}\rho \vec{v}^2\right)\vec{v}}_{\text{does not receive corrections}} + \underbrace{\underline{\underline{\Pi}} \cdot \vec{v}}_{\text{contribution from momentum flux in LRF}} + \underbrace{\vec{J}_q}_{\text{affinity dependent flux in LRF}}$$

Next we need the conservation laws which give us dynamical equations of motion

particle number:

$$\frac{\partial n}{\partial t} + \vec{\nabla} \vec{J}_n = 0 \quad \text{continuity equation}$$

using

$$\vec{J}_n = n\vec{v} \qquad \overset{\rho = m n}{} \qquad \boxed{\frac{\partial \rho}{\partial t} + \vec{\nabla}\left(\rho \vec{v}\right) = 0}$$

which expresses mass conservation

Similarly we need to consider momentum conservation

$$\underbrace{\frac{\partial}{\partial t}\left(\rho v^{\alpha}\right)}_{\text{momentum density } \alpha} + \underbrace{\frac{\partial}{\partial x^{\beta}} J_{p}^{\alpha\beta}}_{\text{divergence of momentum flux } \alpha} = 0 \qquad \left(\sum_{\beta} \text{implicit}\right)$$

So general form

$$\frac{\partial}{\partial t}\left(\rho v^{\alpha}\right) + \frac{\partial}{\partial x^{\beta}}\left(\Pi^{\alpha\beta} + \rho v^{\alpha} v^{\beta}\right) = 0$$

using cont. eqn for ρ

$$\rho\left(\frac{\partial v^{\alpha}}{\partial t} + \underbrace{v^{\beta}\frac{\partial}{\partial x^{\beta}}}_{(\vec{v}\cdot\vec{\nabla})} v^{\alpha}\right) + \frac{\partial}{\partial x^{\beta}} \Pi^{\alpha\beta} = 0$$

finally for the energy density

$$\frac{\partial}{\partial t}\left(\varepsilon + \frac{1}{2}\rho\vec{v}^2\right) + \vec{\nabla}\vec{J}_E = 0$$

$$\frac{\partial}{\partial t}\left(\varepsilon + \frac{1}{2}\rho\vec{v}^2\right) + \frac{\partial}{\partial x^\alpha}\left(\left(\varepsilon + \frac{1}{2}\rho\vec{v}^2\right)v^\alpha + \Pi^{\alpha\beta}v^\beta + J_u^\alpha\right) = 0$$

can be re-arranged using EOM for ρ, ρv^α

$$\frac{\partial}{\partial t}\varepsilon + \frac{\partial}{\partial x^\alpha}\left(\varepsilon v^\alpha + J_u^\alpha\right) + \Pi^{\alpha\beta}\frac{\partial v^\beta}{\partial x^\alpha}$$

$$+ \underbrace{\frac{\partial}{\partial t}\left(\frac{1}{2}\rho\vec{v}^2\right) + \frac{\partial}{\partial x^\alpha}\left(\frac{1}{2}\rho\vec{v}^2 v^\alpha\right) + v^\beta\frac{\partial}{\partial x^\alpha}\Pi^{\alpha\beta}}_{= (1)} = 0$$

Need to simplify

$$\frac{\partial}{\partial t}\left(\frac{1}{2}\rho\vec{v}^2\right) = \frac{\partial}{\partial t}\left(\frac{1}{2}\rho v^\beta v^\beta\right) = v^\beta\frac{\partial}{\partial t}\left(\rho v^\beta\right) - \frac{1}{2}\vec{v}^2\left(\frac{\partial}{\partial t}\rho\right)$$

$$\frac{\partial}{\partial x^\alpha}\left(\frac{1}{2}\rho\vec{v}^2 v^\alpha\right) = \frac{\partial}{\partial x^\alpha}\left(\frac{1}{2}\rho v^\beta v^\beta v^\alpha\right)$$

$$= v^\beta\frac{\partial}{\partial x^\alpha}\left(\rho v^\beta v^\alpha\right) - \frac{1}{2}\vec{v}^2\frac{\partial}{\partial x^\alpha}\left(\rho v^\alpha\right)$$

with this

$$(1) = v^\beta\left[\frac{\partial}{\partial t}\left(\rho v^\beta\right) + \frac{\partial}{\partial x^\alpha}\left(\Pi^{\alpha\beta} + \rho v^\alpha v^\beta\right)\right]$$

$$- \frac{1}{2}\vec{v}^2\left(\frac{\partial}{\partial t}\rho + \frac{\partial}{\partial x^\alpha}\left(\rho v^\alpha\right)\right)$$

which vanishes identically by EOM for ρ, ρv^α

Collecting the remaining terms in ⊗

$$\boxed{\frac{\partial}{\partial t}\mathcal{E} + \frac{\partial}{\partial x^\alpha}\left(\mathcal{E} v^\alpha + J_q^\alpha\right) + \Pi^{\alpha\beta}\frac{\partial v^\beta}{\partial x^\alpha} = 0}$$

from which kinetic energy $\frac{1}{2}\rho \vec{v}^2$ has completely disappeared

$\rightarrow$ accounted for in density and momentum density evolution equations

Since in this case we already have equilibrium fluxes, lets first consider the case where fluxes take equilibrium values

$$J_p^{\alpha\beta} = \Pi^{\alpha\beta} = J_p^{eq\,\alpha\beta} = P\delta^{\alpha\beta}$$

$$\vec{J}_u = \vec{J}_u^{eq} = 0$$

then

Conservation of mass: "change in time = div flow"

Ideal NR Hydrodynamics

$$\frac{\partial}{\partial t}\rho + \vec{\nabla}(\rho\vec{v}) = 0$$

$$\rho\left[\frac{\partial\vec{v}}{\partial t} + (\vec{v}\cdot\vec{\nabla})\vec{v}\right] = \underbrace{-\vec{\nabla}P}$$

"mass x acceleration" "transport of velocity" "Force"

$$\frac{\partial e}{\partial t} + \vec{\nabla}(e\vec{v}) = -P(\vec{\nabla}\cdot\vec{v})$$

change of internal energy due to work done by pressure

In particular for incompressible fluid (ρ = const)

continuity eqn $\Rightarrow$ $\boxed{\vec{\nabla}\vec{v} = 0}$

$$D(\rho\vec{v}) = \rho\left(\frac{\partial\vec{v}}{\partial t} + (\vec{v}\cdot\vec{\nabla})\vec{v}\right) = -\vec{\nabla}P$$

$$De = \frac{\partial e}{\partial t} + (\vec{v}\cdot\vec{\nabla})e = 0$$

$$D = \frac{\partial}{\partial t} + \vec{v}\cdot\vec{\nabla} \quad \text{convective derivative}$$

Now we consider non-equilibrium corrections to ideal hydrodynamics

$\Rightarrow$ will modify constitutive relations for currents adding „dissipative" terms

as usual limit to linear Markovian regime

$\Rightarrow$ „Newtonian fluids"

Generally speaking we have $\vec{J}_N, \vec{J}_e, \underline{\underline{\vec{J}}}_{\vec{p}}$ fluxes | $\vec{\nabla}\left(\frac{1}{T}\right), \vec{\nabla}\left(\frac{-\mu}{T}\right), \vec{\nabla}\left(\frac{-\vec{v}}{T}\right)$ affinities

15 fluxes 15 affinities

$\Rightarrow$ general L_{ab} is 15×15 matrix

However by definition of velocity ~~can eliminate~~

$$\vec{J}_N = \vec{J}_N^{eq} = n\vec{v}$$

can eliminate $L_{Na} = 0 \underset{\text{Onsager relation}}{\Rightarrow} L_{aN} = 0$

Gradients in chemical potential do not lead directly to dissipation.
However $\vec{\nabla}\left(\frac{-\mu}{T}\right)$ will lead to variations of pressure P, which leads to variations of velocity $\vec{v}$
So physical effects are still included

By virtue of Curie principle can not couple

$\vec{\nabla}\left(\frac{1}{T}\right)$ vector

with $\vec{J}_{\vec{p}}$ tensor

as they behave differently under parity

same for

$\vec{\nabla}\left(\frac{\vec{v}}{T}\right)$ tensor

and $\vec{J}_E$ vector

So we are left with

$$J_q^{\alpha} = L_{EE}^{\alpha\beta} \frac{\partial}{\partial x^{\beta}}\left(\frac{1}{T}\right)$$

$$J_{\vec{p}}^{\alpha\beta} = L_{\vec{p}\vec{p}}^{\alpha\beta\gamma\delta} \frac{\partial}{\partial x^{\delta}}\left(-\frac{v^{\gamma}}{T}\right)$$

(Summation over repeated indices implied)

in particular in LRF $\frac{\partial}{\partial x^{\delta}}\left(-\frac{v^{\gamma}}{T}\right) = -\frac{1}{T}\frac{\partial v^{\gamma}}{\partial x^{\delta}}$

so since dissipative terms only describe modifications of constitutive relations in LRF we conclude

$$J_{\vec{p}}^{\alpha\beta} = \underbrace{-\frac{1}{T} L_{\vec{p}\vec{p}}^{\alpha\beta\gamma\delta}}_{=\tilde{L}_{\vec{p}\vec{p}}^{\alpha\beta\gamma\delta}} \frac{\partial v^{\gamma}}{\partial x^{\delta}}$$

Next we can take into account rotational
Symmetry in LRF, which for energy response
simply means

$$\underbrace{L^{\alpha\beta}_{EE} = L_{EE}\,\underbrace{\delta^{\alpha\beta}}_{\substack{\text{rotationally invariant}\\ \text{rank 2 tensor}}}} \Rightarrow \text{1 single coefficient}$$

Similarly for momentum response

$$L^{\alpha\beta\gamma\delta}_{\vec{p}\vec{p}} = C_1\,\delta^{\alpha\beta}\delta^{\gamma\delta} + C_2\,\delta^{\alpha\gamma}\delta^{\beta\delta} + C_3\,\delta^{\alpha\delta}\delta^{\beta\gamma}$$

Since $J^{\alpha\beta}_{pp} = \Pi^{\alpha\beta} = P\delta^{\alpha\beta} - \frac{1}{T}L^{\alpha\beta\gamma\delta}\frac{\partial v^\gamma}{\partial x^\delta}$

and $\Pi_{\alpha\beta} = \Pi_{\beta\alpha}$ to ensure angular momentum conservation

$\Rightarrow C_2 = C_3$

So two independent coefficients
Conventionally, one uses the tensor
structures

$$\delta^{\alpha\beta}\delta^{\gamma\delta}\ ,\ \frac{1}{2}\left(\delta^{\alpha\gamma}\delta^{\beta\delta} + \delta^{\alpha\delta}\delta^{\beta\gamma}\right) - \frac{1}{3}\delta^{\alpha\beta}\delta^{\gamma\delta}$$

Such that

$$\Pi^{\alpha\beta} = P\,\delta^{\alpha\beta} - \frac{L^{(1)}_{\vec{p}\vec{p}}}{T}\,\delta^{\alpha\beta}\underbrace{\frac{\partial v^\gamma}{\partial x^\gamma}}_{\vec{\nabla}\cdot\vec{v}} - \frac{L^{(2)}_{\vec{p}\vec{p}}}{T}\,\sigma^{\alpha\beta}$$

where σ is the shear stress tensor

$$\sigma^{\alpha\beta} = \frac{1}{2}\left(\frac{\partial v_\alpha}{\partial x^\beta} + \frac{\partial v_\beta}{\partial x^\alpha}\right) - \frac{1}{3}\left(\vec{\nabla}\cdot\vec{v}\right)\delta^{\alpha\beta}$$

$$\mathrm{tr}\left[\underline{\underline{\sigma}}\right] = 0$$

So in summary we have 3 kinetic coefficients

$$\underline{\underline{\Pi}} = \left(P - \frac{L^{(1)}_{PP}}{T}\right)\underline{\underline{1}} - \frac{L^{(2)}_{PP}}{T}\underline{\underline{\sigma}}$$

$$\vec{J}_u = L_{EE}\,\vec{\nabla}\left(\frac{1}{T}\right)$$

which are related as

$$\vec{J}_q = -\mathcal{K}\,\vec{\nabla}T \qquad \mathcal{K} = \frac{L_{EE}}{T^2}$$ „heat conductivity"

$$\frac{L^{(1)}_{PP}}{T} = \zeta$$ „bulk viscosity"

$$\frac{L^{(2)}_{PP}}{2T} = \eta$$ „shear viscosity"

so

$$\underline{\underline{\Pi}} = \left(P - \zeta\left(\vec{\nabla}\cdot\vec{v}\right)\right)\underline{\underline{1}} - 2\eta\,\underline{\underline{\sigma}}$$

Now how does this affect EOMs

$$① \quad \rho\left[\frac{\partial v^\alpha}{\partial t} + (\vec{v}\cdot\vec{\nabla})v^\alpha\right] + \frac{\partial}{\partial x^\beta}\Pi^{\alpha\beta} = 0$$

assuming derivatives of η, ζ to be negligible

$$\boxed{① \quad \rho\left[\frac{\partial \vec{v}}{\partial t} + (\vec{v}\cdot\vec{\nabla})\vec{v}\right] = -\vec{\nabla}P + \zeta\vec{\nabla}(\vec{\nabla}\cdot\vec{v}) + \eta\left(\Delta\vec{v} + \frac{1}{3}\vec{\nabla}(\vec{\nabla}\cdot\vec{v})\right)}$$

Navier-Stokes Equation

$$\frac{\partial}{\partial x^\beta}\sigma^{\alpha\beta} = \frac{\partial}{\partial x^\beta}\frac{1}{2}\left(\frac{\partial v^\alpha}{\partial x^\beta} + \frac{\partial v^\beta}{\partial x^\alpha}\right) - \frac{1}{3}(\vec{\nabla}\cdot\vec{v})\delta^{\alpha\beta}$$

$$= \frac{1}{2}\frac{\partial^2 v^\alpha}{\partial x_\beta \partial x^\beta} + \left(\frac{1}{2} - \frac{1}{3}\right)\frac{\partial^2 v^\beta}{\partial x_\alpha \partial x^\beta}$$

in addition to the hydrostatic pressure term $-\vec{\nabla}P$ there are additional friction forces