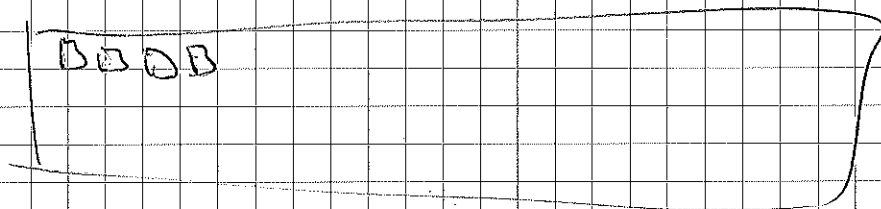


## 1.3 Local equilibrium of continuous systems

In reality we are interested in describing continuous systems of large size (thermodynamic limit)

If system as a whole is sufficiently large, may divide into arbitrary many sub-cells



such that

- extensive quantities  $X_i^{(U)}$  vary slowly on the scale of a single cell  $\rightarrow$  can be treated as approximately constant
- each cell is large enough to be meaningfully described by thermodynamics

By taking the limit of cell size to 0, we obtain a continuous description, labeled by space-time point  $(t, \vec{r})$

$$X_i^{(U)} \rightarrow \{X_i(t, \vec{r})\} \quad (\text{fields})$$

Now in making the cells small, extensive  
variables  $X_i^{(a)} \rightarrow 0$  become meaningless

$\Rightarrow$  Consider densities instead

Non-relativistic literature:

Essentially two conventions how to define  
densities:

- a)  $w = m$  mass (const  $m$  per cell)
- b)  $w = V$  volume (const  $V$  per cell)

$$b) \Delta V^{(a)} = \Delta V^{(b)} \quad X_i^{(a)}(h, \vec{r}) = \lim_{\Delta V \rightarrow 0} \frac{X_i^{(a)}}{\Delta V^{(a)}}$$

$$a) \Delta M^{(a)} = \Delta M^{(b)} \quad X_i^{(a)}(h, \vec{r}) = \lim_{\Delta M \rightarrow 0} \frac{X_i^{(a)}}{\Delta M^{(a)}}$$

obviously the two are related

$$X_i^{(a)}(h, \vec{r}) = \rho(h, \vec{r}) X_i^{(b)}(h, \vec{r})$$

where  $\rho(h, \vec{r})$  mass density

Classical Entropy of local equilibrium system  
 is given by Gibbs relation

$$dS(t, \vec{r}) = \sum_i Y_i(t, \vec{r}) dx_i(t, \vec{r}) \quad (dx_V = 0 \text{ in volume element})$$

So we obtain the ~~the~~ local intensive variables as

$$Y_i(t, \vec{r}) = \frac{\partial S(t, \vec{r})}{\partial x_i(t, \vec{r})} = \frac{\delta S^{\text{tot}}(t)}{\delta x_i(t, \vec{r})}$$

partial derivative
functional derivative

Similarly it follows that the intensive variable ~~is~~ <sup>sehr</sup>

$$Y_i(t, \vec{r}) = Y_i(x_i(t, \vec{r}))$$

local EOS (which are of the same form as global thermodynamic ones)

Note that its easy to transition from  $X_i$  to  $\Omega_i$  variables. e.g. ideal gas

$$k_B T = \frac{2}{3} \frac{U}{N} = \frac{2}{3} \frac{e}{h}$$

# Criteria for Validity of local equilibrium assumption:

1) Fluctuations need to be small enough compared to mean values for TD to be meaningful



$$\frac{S_{fl}}{Nk_B} \ll 1$$

if  $l$  denotes the interparticle distance this translates into a condition

$$l \gg d$$

2) Variations of the thermodynamic quantities have to be sufficiently slow to not drive the system out-of-equilibrium

Since equilibrium is restored by interparticle interactions, the relevant microscopic scale is  $l_{int}$  (mean-free path)

In order to maintain approx. local equilibrium in the presence of gradients

$$l \gg l_{int} \ll l$$

$$\left( \nabla X \sim \frac{X}{L_{macro}} \right)$$

## 1.4 Alternatives to fluxes in continuous media

In the discussion of entropy production in discrete systems, we frequently made use of conservation laws.

We will now formulate the same ideas for cont. media

### Balance equations:

Consider subvolume  $V$  bounded by  $\partial V$  of the system.

Scalar amount of extensive quantity  $G(t)$  contained in  $V$  at a given time

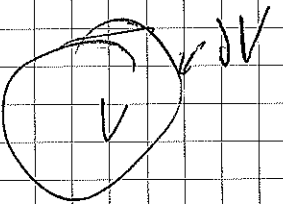
is

$$G(t) = \int_V g(t, \vec{r}) d^3\vec{r} = \int_V \rho(t, \vec{r}) g_{(m)}(t, \vec{r}) d^3\vec{r}$$

Now we are interested how this may change in time

two possibilities:

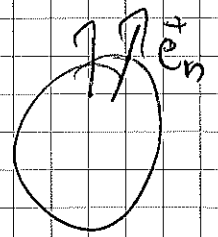
- a) flux in/out of volume  $V$
- b) local production of  $G$  inside  $V$



a) Given by flux through surface

$$\delta G_{(a)} = \int_G(t, \vec{r}) \cdot \vec{e}_n(\vec{r}) d^2 r_s dt$$

current density /  
flux density



b) Given by

$$\delta G_{(b)} = \rho_G(t, \vec{r}) d^3 r dt$$

Source density

So

$$\frac{dG(t)}{dt} + \int_{dV} \int_G(t, \vec{r}) \vec{e}_n(\vec{r}) d^2 r_s = \int_{dV} \rho_G(t, \vec{r})$$

Since the volume is fixed  
can remark

$$\frac{dG(t)}{dt} = \int_{dV} \frac{d\rho_G(t, \vec{r})}{dt}$$

Using Gauss theorem for the second  
term we then have

Gauss theorem  $\int_{\partial V} \vec{J}_G \cdot \vec{e}_n d^2r = \int_V \vec{\nabla} \cdot \vec{J}_G d^3r$

$$\Rightarrow \int_V d^3r \left( \frac{dG(\vec{r}, t)}{dt} + \vec{\nabla} \cdot \vec{J}_G(\vec{r}, t) - \mathcal{C}_G(\vec{r}, t) \right) = 0$$

Since this is true for any  $V$  we obtain

local balance equation:

$$\boxed{\frac{dG(\vec{r}, t)}{dt} + \vec{\nabla} \cdot \vec{J}_G(\vec{r}, t) = \mathcal{C}_G(\vec{r}, t)}$$

Conserved quantities:

In the special case  $\mathcal{C}_G(\vec{r}, t) \equiv 0$  for  $G = \text{conserved}$   
 this is called continuity equation or  
local conservation law.

In the case of entropy

$$\frac{dS}{dt} + \int_{\partial V} \underbrace{\vec{J}_S \cdot \vec{e}_n}_{\text{entropy current}} d^2r = \int_V \mathcal{C}_S d^3r$$

$$\Rightarrow \frac{dS}{dt} = \frac{dS_{\text{exch}}}{dt} + \frac{dS_{\text{intprod}}}{dt}$$

↑ entropy production rate

The entropy production source term is  
 positive semi-definite  $\mathcal{C}_S \geq 0$  (positive contributions  
 are called dissipative phenomena)

Entropy current, affinity & fluxes:

Based on Gibbs relation, we also find

$$(1) \quad \frac{dS}{dt}(\vec{r}, \vec{r}') = \sum_i Y_i(\vec{r}, \vec{r}') \frac{dX_i(\vec{r}, \vec{r}')}{dt} \quad \left( \frac{dX_i}{dt} = 0 \right)$$

characterizing static change of the entropy density.

Considering that we could also look at a moving system (or a static system in a moving frame) and that the IR description should be Galilean invariant, this suggests that the currents satisfy a similar relation

$$(2) \quad \vec{J}_S(\vec{r}, \vec{r}') = \sum_i Y_i(\vec{r}, \vec{r}') \vec{J}_i(\vec{r}, \vec{r}') \quad (\vec{J}_i = 0)$$

where  $\vec{J}_i$  is flux density. We can then combine (1) and (2) to calculate entropy production rate

$$\begin{aligned} \sigma_S(\vec{r}, \vec{r}') &= \frac{dS}{dt}(\vec{r}, \vec{r}') + \vec{\nabla} \cdot \vec{J}_S(\vec{r}, \vec{r}') \\ &= \sum_i \left( \vec{\nabla} Y_i(\vec{r}, \vec{r}') \right) \cdot \vec{J}_i(\vec{r}, \vec{r}') \end{aligned}$$

where in the last step we used the continuity equation



In analogy to the discrete systems  
we can then define

$$\vec{F}_i \equiv \vec{\nabla} \chi_i(t, \vec{r})$$

"affinity"

and the entropy production rate is  
again given by

$$\sigma(t, \vec{r}) = \sum_i \vec{F}_i(t, \vec{r}) \cdot \vec{J}_i(t, \vec{r})$$

Note that differences (discrete systems) have now  
become derivatives in continuous systems.

Note also we have only considered scalar  
quantities (energy, particle number, ...) so far  
for those the fluxes and affinities are  
vectors

We will discuss later (Nussenzweig - Stokes Hydro) how  
in the case of e.g. the vector  $\vec{P}$ , the  
affinities and fluxes become tensors

However Bittman structure remains fully general