Will formalize stochastic process a little more to get different view on Brownian motion

So far we always looked at individual microscopic processes where $v(t)$ is known and subsequently performed averages over all possible microscopic realizations

Instead will we look at ensemble of all possible microscopic realizations

probability distribution $p_1(t_1, v)$

such that $\langle v(t) \rangle = \int dv \, p_1(t, v) \, v$

more generally $p_n(t_1, v_1, \dots, t_n, v_n)$

$$\langle v(t_1) \dots v(t_n) \rangle = \int dv_1 \dots dv_n \, (v_1 \dots v_n) \, p_n(t_1, v_1, \dots, t_n, v_n)$$

analogous to n-body phase space distributions

$$p(t_1, v_1, \dots t_{n-1}, v_{n-1}) = \int dv_n \, p_n(t_1, v_1, \dots, t_n, v_n)$$

Now in addition to probability distributions p_n can also define conditional probability distributions

$$P_{m|n-m}(t_1, v_1, \ldots, t_m, v_m | t_{m+1}, v_{m+1}, \ldots t_n, v_n)$$

which gives probability for $t_1, v_1 \ldots t_m, v_m$ knowing $t_{m+1}, v_{m+1}, \ldots t_n, v_n$

$$= \frac{P_n(t_1, v_1 \ldots t_n, v_n)}{P_{n-m}(t_{m+1}, \ldots, t_n, v_n)}$$

likelihood that all values are there / likelihood that $m+1 \ldots$ are values

Likewise can construct p_n as

$$P_n = P_{m|n-m} \, P_{n-m}$$

Now it turns out that there is a particular class of processes called Markov processes which are particularly useful for many applications

Basically means that evolution of stochastic process has no memory on its past

for $t_1 < t_2 < \dots$ t_n we have

$$P_{1|n}(t_{n+1}, v_{n+1} | t_1, v_1 \dots t_n, v_n) = P_{1|1}(t_{n+1}, v_{n+1} | t_n, v_n)$$

So conditional probability for transitioning from $t_n, v_n \Rightarrow t_{n+1}, v_{n+1}$ only depends on current state of the system

Important consequence all of process is determined by single probability P_1 and transition probability $P_{1|1}$ namely

$$P_n(t_1, v_1, \dots t_n, v_n) = P_{1|1}(t_n, v_n | t_{n-1}, v_{n-1}) \dots P_{1|1}(t_2, v_2 | t_1, v_1) P_1(t_1, v_1)$$

So if we consider

$$P_{1|1}(t_2, v_2 | t_1, v_1) = \frac{P_2(t_1, v_1, t_2, v_2)}{P_1(t_1, v_1)}$$

$$= \frac{\int dv_2 \, P_3(t_1, v_1, t_2, v_2, t_3, v_3)}{P_1(t_1, v_1)}$$

$$= \int dv_2 \, P_{1|1}(t_3, v_3 | t_2, v_2) \, P_{1|1}(t_2, v_2 | t_1, v_1)$$

which is the Chapman - Kolmogorov equation

Now how does this help us to study Brownian motion

Strictly speaking it is not: $F_L(t)$ is not white. We have finite auto correlation time so

Brownian motion is not a Markov process

However if we look at scales $\Delta t \gg \tau_{corr}$ then within this coarse grained description we encounter independent collisions and can treat process as Markov

Then what we want to know is

$P_n(t_1 v_1, \ldots, t_n v_n)$ which for a Markov process greatly simplifies knowing $P_{1|1}$ and P_1

Now how do we do this

$$P_1(t+\Delta t, v) = \int dv' \, P_{1|1}(t+\Delta t, v \mid t, v') \, P_1(t, v')$$

with $\Delta t \gg \tau_{corr}$

Can recast into integration over

$w = v - v'$

$$P_1(t+\Delta t, v) = \int dw\, P_{1|1}(t+\Delta t, v+w-w \mid t, v-w)\, P_1(t, v-w)$$

Now is now for $\Delta t \ll \frac{1}{\gamma}$ that we can expand RHS into Taylor series in w assuming that velocity changes are typically small

$$= \sum_{n=0}^{\infty} \int dw \frac{(-w)^n}{n!} \frac{\partial^n}{\partial v^n} \left[P_{1|1}(t+\Delta t, v+w \mid t, v)\, P_1(t, v) \right]$$

Now the only w dependence is in $P_{1|1}$

So define moments

$$M_n(t, t+\Delta t, v) = \int dw\, w^n\, P_{1|1}(t+\Delta t, v+w \mid t, v)$$

We have for $n=0$ $\boxed{M_0 = 1}$ since probability is conserved

While for in the limit $\Delta t \to 0$ we will assume to be of the form

$$\lim_{\Delta t \to 0} M_n(t, t+\Delta t, v) = \mu_n(t, v)\, \Delta t \qquad n \geq 1$$

i.e. changes are linear in Δt

So plugging in we get Kramers-Moyal expansion

$$\frac{P_1(t+\Delta t, v) - P_1(t,v)}{\Delta t} = \sum_{n=1}^{\infty} \frac{(-1)^n}{n!} \frac{\partial^n}{\partial v^n} M_n(t,v) P_1(t,v)$$

in particular if only the two lowest order moments contribute we obtain in the limit $\Delta t \to 0$

$$\frac{\partial}{\partial t} P_1(t,v) = - \frac{\partial}{\partial v} \left[M_1(t,v) \, P_1(t,v) \right] + \frac{1}{2} \frac{\partial^2}{\partial v^2} \left[M_2(t,v) \, P_1(t,v) \right]$$

which is called the Fokker-Planck equation

Now let's compute the moments for Brownian motion

$$v(t+\Delta t) = -\gamma \int_t^{t+\Delta t} v(t')\,dt' + \frac{1}{m}\int_t^{t+\Delta t} F_L(t')\,dt'$$

So looking at $M_1 = \langle v(t+\Delta t) - v(t) \rangle$

$$= -\gamma\, v(t)\,\Delta t + O((\gamma \Delta t)^2)$$

Whereas $M_2 = \langle [v(t+\Delta t) - v(t)]^2 \rangle$

$$= \left(-\gamma\, v(t)\,\Delta t\right)^2 - \frac{2\gamma}{m}\int_t^{t+\Delta t} dt' \int_t^{t+\Delta t} dt'' \langle v(t') F_L(t'') \rangle$$

$$+ \frac{1}{m^2}\int_t^{t+\Delta t} dt' \int_t^{t+\Delta t} dt'' \langle F_L(t') F_L(t'') \rangle$$

Now since $\Delta t \gg \tau_{coll}$ we can approximate autocorrelation function by white noise:

$$= 2 D_v \Delta t + O(\Delta t^2)$$

So for Brownian motion we get

$$\boxed{\frac{\partial f}{\partial t}(t,v) = \gamma \frac{\partial}{\partial v}\left[v f(t,v) \right] + D_v \frac{\partial^2 f}{\partial v^2}(t,v)}$$

where $f(t,v) = P_1(t,v)$

Can directly interpret terms on RHS as attributed to drag and diffusion

Since FPE is evolution equation for probability expect conservation of total probability to find particle with velocity v to manifest itself somehow

Inspecting RHS, can rewrite

$$\frac{\partial f(t,v)}{\partial t} = \frac{\partial}{\partial v}\left[\gamma v f(t,v) + D_v \frac{\partial}{\partial v} f(t,v)\right]$$

$$\Rightarrow \frac{\partial f(t,v)}{\partial t} + \frac{\partial}{\partial v} J_v(t,v) = 0$$

where $J_v(t,v) = -\gamma v f(t,v) - D_v \frac{\partial}{\partial v} f(t,v)$

is probability current in velocity space

Based on this decomposition becomes straightforward to construct equilibrium solution based on

$$J_v(t,v) = 0 \quad \Rightarrow \quad \frac{\partial f}{\partial t}$$

so as find

$$\frac{\partial}{\partial v} f_{eq}(t,v) = -\frac{\gamma}{D_v} v f_{eq}(t,v)$$

$$\Rightarrow f_{eq}(t,v) = \sqrt{\frac{\gamma}{2\pi D_v}}\, e^{-\frac{\gamma}{2D_v} v^2}$$

where constant is determined by normalization of probability

Now lets look at time dependent solution of FPE

Since FPE is linear PDE solution can be constructed in terms of linear superposition of elementary solutions

Will consider simplest case $f(t=0, v) = \delta(v - v_0)$ and construct solution in Fourier space wrt velocity

$$\tilde{f}(t,k) = \int_{-\infty}^{+\infty} dv \; f(t,v)\, e^{ikv}$$

$$\tilde{f}(t=0,k) = e^{ikv_0}$$

Now lets look at Fokker-Planck equation

$$v = -i\frac{\partial}{\partial k} \qquad \frac{\partial}{\partial v} = -ik$$

$$\frac{\partial}{\partial t}\tilde{f}(t,k) = \left(-D_v k^2 - \gamma k \frac{\partial}{\partial k}\right)\tilde{f}(t,k)$$

more technically we have

$$\text{as} \quad \gamma \int_{-\infty}^{+\infty} dv \left(\frac{\partial}{\partial v} v f(t,v)\right) e^{ikv}$$

$$= \gamma \int_{-\infty}^{+\infty} dv \frac{\partial}{\partial v}\Big(\qquad\Big) - ik\gamma \int_{-\infty}^{+\infty} dv \; v f(t,v)\, e^{ikv}$$

$$= -k\frac{\partial}{\partial k}\tilde{f}(t,k)$$

Now we have transformed second order PDE
into first order PDE

$$\left[\frac{\partial}{\partial t} + \gamma k \frac{\partial}{\partial k}\right] \tilde{f}(t,k) = -D_v k^2 \tilde{f}(t,k)$$

for which solution is not obvious

However for this kind of PDE can solve
by method of characteristics

Basic idea is to introduce fictitious variable s
and rewrite LHS as

$$\frac{d}{ds}\tilde{f}(t(s),k(s)) \quad \text{where now } t(s) \text{ and } k(s) \text{ depend on } s$$

$$= \frac{dt(s)}{ds}\frac{\partial \tilde{f}}{\partial t}(t(s),k(s)) + \frac{dk(s)}{ds}\frac{\partial \tilde{f}}{\partial k}(t(s),k(s))$$

So if we identify

$$\frac{dt}{ds} = 1 \quad \Rightarrow \quad s = t$$

and

$$\frac{dk(s)}{ds} = \gamma k(s) \quad \Rightarrow \quad k(s) = k(s=0)\, e^{\gamma s}$$

then PDE becomes

$$\frac{d}{ds}\tilde{f}(t(s),k(s)) = -D_v k(s)^2 \tilde{f}(t(s),k(s))$$

Since we know $k(s)$ can solve directly

$$\tilde{f}(t(s), k(s)) = \exp\left(-D_v \int_0^s ds'\, k^2(s=0)\, e^{2\gamma s'}\right) \tilde{f}(t(s=0), k(s=0))$$

$$= \exp\left(-\frac{D_v k^2(s=0)}{2\gamma}\left(e^{2\gamma s}-1\right)\right) \tilde{f}(t(s=0), k(s=0))$$

where $t(s)=t \quad k(s)=k(s=0)\, e^{\gamma s}$

So expressing $k(s=0)=k(s)\, e^{-\gamma s}$ to invert the solution we find

$$\tilde{f}(t,k) = \exp\left(-\frac{D_v k^2}{2\gamma}\left(1-e^{-2\gamma t}\right)\right) \tilde{f}(t=0, k e^{-\gamma t})$$

$$= \exp\left(-\frac{D_v k^2}{2\gamma}\left(1-e^{-2\gamma t}\right)\right) e^{ikv_0 e^{-\gamma t}}$$

Now all that's left to be done is to perform inverse FT

$$f(t,v) = \int_{-\infty}^{\infty} \frac{dk}{2\pi} \tilde{f}(t,k)\, e^{-ikv}$$

$$= \sqrt{\frac{\gamma}{2\pi D_v (1-e^{-2\gamma t})}} \exp\left(-\frac{\gamma}{2D_v} \frac{(v-v_0 e^{-\gamma t})^2}{1-e^{-2\gamma t}}\right)$$

$\Rightarrow$ Gaussian probability distribution with mean

$$\langle v(t)\rangle = v_0 e^{-\gamma t}$$

$$\sigma^2_{v(t)} = \frac{D_v}{\gamma}\left(1-e^{-2\gamma t}\right)$$

in line with our analysis of stochastic equation

Diffusion in phase space described by SDE for two variables

$$\frac{d}{dt}\begin{pmatrix} x(t) \\ v(t) \end{pmatrix} = \begin{pmatrix} v(t) \\ -\gamma v(t) + \frac{1}{M} F_{ext}(t) \end{pmatrix} + \begin{pmatrix} 0 \\ \frac{1}{M} F_L(t) \end{pmatrix}$$

where the first equation is in fact deterministic

Probability distribution $p_1(t, x, v) = f(t, x, v)$

So following the same steps as previously we can derive the corresponding FPE

$$\left(\frac{\partial}{\partial t} + v\frac{\partial}{\partial x} + \frac{F_{ext}}{M}\frac{\partial}{\partial v}\right) f(t, x, v) = \gamma \frac{\partial}{\partial v}\left[v f(t, x, v) + D_v \frac{\partial}{\partial v} f(t, x, v)\right]$$

which is called Kramers-Chandrasekhar equation

Similar version of BTE, where collision integral is approximation by diffusion approximation