Recap:

We finished discussion of stationary turbulence

Showed again as were told that 3-wave kinetic equations features stationary solutions, defined by the condition

$$\frac{\partial}{\partial t} n(k,t) = I[n](k,t) = 0$$

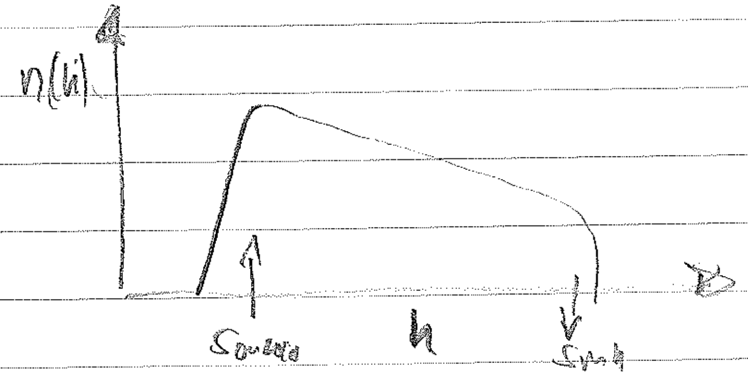
within our inertial range of wavenumbers

where

$$n(k,t) = n_{KZ}(k) = n_0 \left(\frac{k}{k_0}\right)^{s_0}$$

with $S_0 =$ ~~and~~

In order to better understand the physical significance of these solutions, we considered balance equations in k-space



Globally $\frac{dE}{dt} = \dot{E}_{in} - \dot{E}_{out} \overset{!}{=} 0$

and $\dot{E}_{in} = $ const as a function of time

Differentially in k, the energy injected at k_{source} needs to be transported to k_{sink} where it is removed from the system

Energy transport within a inertial range of cascade is described by continuity equation in k-space for

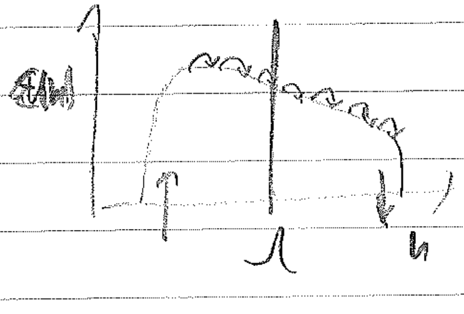
$$\mathcal{E}(k,t) = \omega(k)\, n(k,t)$$

which reads

$$\frac{\partial}{\partial t} \mathcal{E}(k,t) + \vec{\nabla}_k \vec{J}_{\mathcal{E}}(k,t) = 0$$

where $\vec{\nabla}_k \vec{J}_{\mathcal{E}}(k,t) = -\omega(k)\, I[n](k,t)$

Now the important thing to realise is that energy needs to be transported from k_{source} to k_{sink} without accumulation at intermediate scales



So if we look at energy flux from $k<\Lambda$ to $k>\Lambda$ this should be independent of Λ and equal to the energy injection rate $\dot{E}_{in}$ in the stationary turbulent regime

Note that the concept of stationary flux requires the notion of locality of interactions, i.e. energy has to flow through all intermediate scales and can not be transmitted directly to the ends

$\Rightarrow$ necessary condition for realization of KZ spectra

Now assuming this is the case we can investigate what condition this implies

$$J_k(\Lambda,t) = \underbrace{\int_{\partial\Lambda}}_{\text{momentum shell}} \vec{J}_\varepsilon(k,t)\, d\vec{\sigma}_k = -\int^{\Lambda} d^dk \; \omega(k)\, I[n](k,t)$$

By exploiting scale invariance of the interaction

$$J_k(\Lambda,t) = -\Omega^{(d)} \Lambda^{d+m+z} K_0^{-(m+z)} \omega(k_0) \frac{I[n](k_0)}{d+m+z}$$

$\Rightarrow$ Scale invariance is realized for

$d+m+z=0 \quad \Rightarrow \quad n(k) = n_0 \left(\frac{k}{k_0}\right)^{-s_0}$

with $s_0 = m+d$

Now this also shows that even though $I[n](k_0)=0$

$$J_k^S = \lim_{\mathrm{dt}\to 0^+} J_k(\ldots t) = -\Omega^{(d)} k_0^d \, \omega(k_0) \lim_{\mathrm{dt}\to 0^+} \frac{I[n](k_0)}{\mathrm{dt}}$$

the flux remains non-vanishing and the KZ spectrum is therefore associated with a finite flux of a conserved quantity

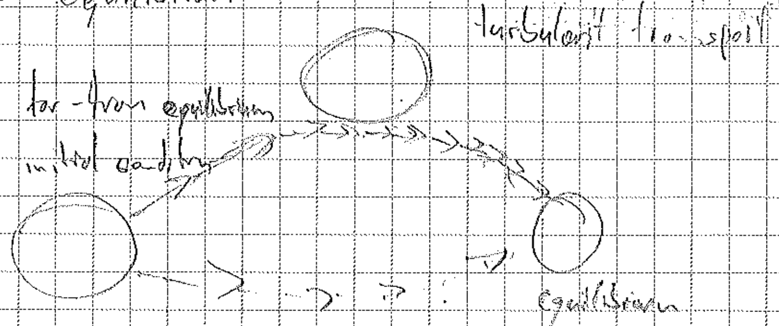
By matching $J_k^S = \dot{E}_{in}$ can determine not only the universal scaling exponent. So but also the non-universal amplitude n_0 of the KZ spectrum

So far we discussed stationary turbulent solutions whose realization requires source & sink e.g. energy injection & dissipation

However the phenomenon of turbulence as defined by transport of conserved quantity between widely separated scales is of relevance for a larger class of problems, also in closed/isolated systems where there is no source or sink present

Key difference in absence of source/sink is that dynamics are not expected to be stationary

General picture: Decaying turbulence can exist for some time before system eventually relaxes towards equilibrium



Clearly, whether or not and for how long decaying turbulence is realized depends on the system at hand

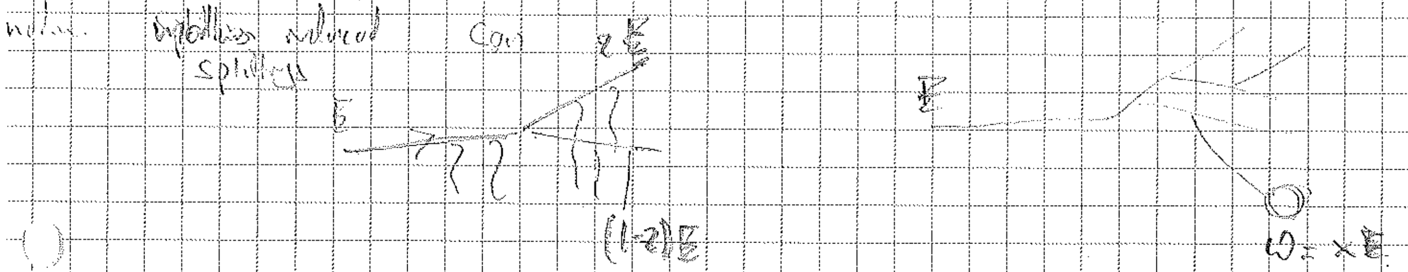
We will consider two examples of energy transport

1) inverse branching cascade
(e.g. Jet energy loss in QCD plasma)

2) non-thermal fixed points in scalar field theory
(e.g. preheating in early universe)

Inverse branching cascade in Jet-energy loss

Jet: Highly energetic particle produced in high-energy hadron collisions



Since splittings are nearly collinear dynamics is effectively 1+1 D

Describe dynamics in terms of

$$D(x,t) = \omega\, n(\omega,t)\Big|_{x=\frac{\omega}{E}} \quad \text{momentum fraction}$$

Since branchings are independent of each other and quasi-instantaneous on the time scale of interest

$\Rightarrow$ Eff. kinetic equation for evolution of $D(x,\tau)$

$$\partial_\tau D(x,\tau) = \int_0^1 dz\, K(z) \left[\sqrt{\frac{z}{x}}\, D\!\left(\frac{x}{z},\tau\right) - \frac{z}{\sqrt{x}}\, D(x,\tau) \right]$$

where symmetries of $K(z)$ have been used to simplify kinetic equation and τ is dimensionless time variable

Can easily check that

$$D(x,\tau) = \frac{c}{\sqrt{x}}$$

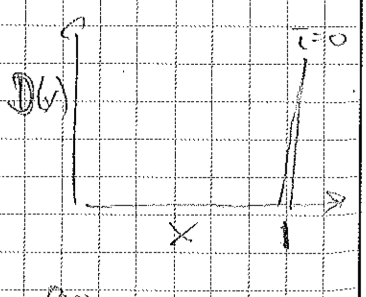
is a stationary KZ solution, with finite energy flux directed towards smaller x

physically: get loss of energy to thermal medium by multiple branchings

Stationary turbulent solution requires source/sink but really we are interested in evolution of a single jet, i.e. no source present in the system

Instead of stationary solution need to consider initial value problem

$$D(x,\tau=0) = \delta(1-x)$$



where initially all energy is carried by one highly energetic particle

Due to [illegible] evolution equation, can find exact analytic solution for specific kernels

$$K(z) \simeq \frac{1}{z^{3/2}(1-z)^{3/2}}$$

$$D(x,\tau) = \frac{\tau}{\sqrt{x}\,(1-x)^{3/2}}\, e^{-\frac{\pi\tau^2}{1-x}}$$

$$\overset{x\ll 1}{\simeq} \frac{\tau}{\sqrt{x}}\, e^{-\pi\tau^2}$$

Kolmogorov spectrum with time dependent amplitude due to decaying source

Inverse Cascade transmits energy from jet $(x \simeq 1)$ to medium $(x \simeq 0)$

Energy carried by jet fragments decreases

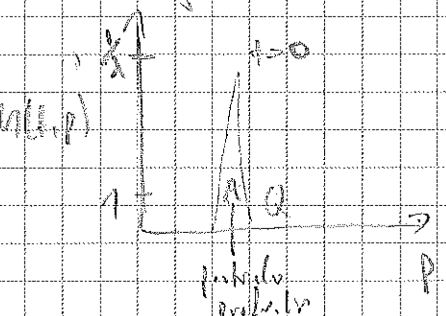
$$E(\tau) = \int_0^1 dx\, D(x,\tau) = e^{-\pi\tau^2}$$

Due to characteristic energy dependence of branching rates

Interestingly energy can be transported across an arbitrarily large separation of scales down to $x \simeq 0$ in a finite amount of time

Non-thermal fixed points in scalar field theory

Ex: Cosmology: Decay of inflaton field leads to production of large number of particles with low momentum



energy density:

$$\epsilon = \int d^3p\, \omega_p\, n(t,p) \sim \frac{Q^4}{\lambda}$$

particle density:

$$n = \int d^3p\, n(t,p) \sim \frac{Q^3}{\lambda}$$

energy per particle

$$\frac{\epsilon}{n} \sim Q$$

Compare with equilibrium

energy $\epsilon_{eq} \sim T_{eq}^4$ $\quad n_{eq} \sim T_{eq}^3$ $\quad \frac{\epsilon_{eq}}{n_{eq}} \sim T_{eq}$

Since energy is conserved during

during thermalization process we can determine

T_{eq} by setting $\epsilon = \epsilon_{eq}$ $\Rightarrow$ $T_{eq} \sim \frac{Q}{\lambda^{1/4}} \overset{\lambda \ll 1}{\gg} Q$

One concludes that for thermalization energy needs to be transported from $p \sim Q$ to $p \sim T$

$\Rightarrow$ (direct) energy cascade

Since $n(t,p) \gg 1$ for typical modes
dynamics governed by kinetic equation
with Bose enhancement (classical wave limit)

Dominant interaction $\sim$ of $1 \leftrightarrow 2$
so for statistically homogeneous & isotropic systems

$$\frac{\partial}{\partial t} n(t,k) = \frac{1}{2} \int d^d k_1 \, d^d k_2 \; \tilde{w}(k \to k_1 k_2) \left[n_1 n_2 - n_k (n_1 + n_2) \right]$$

$$- 2 \, \tilde{w}(k_1 \to k k_2) \left[n_k n_2 - n_1 (n_k + n_2) \right]$$

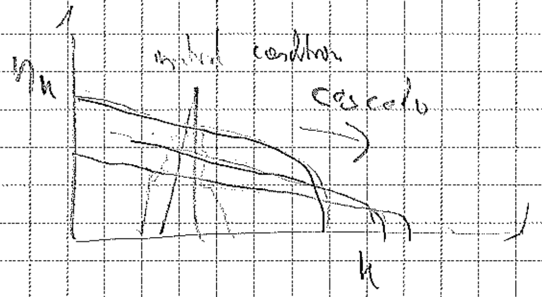
with $\tilde{w}(k \to k_1 k_2) = \frac{36 \lambda^2 \bar{\phi}_0^2}{(2\pi)^2} \delta^{(d)}(k - p_1 - p_2) \frac{\delta(\omega_k - \omega_{p_1} - \omega_{p_2})}{2\omega_k 2\omega_1 2\omega_2}$

for scalar field theory in $d=3$

Now contrary to the example of the branching cascade, it turns out that in this case interaction rate decreases over time along the cascade

e.g. $\Gamma_0 \sim Q$ initially $\qquad \Gamma_{eq} \sim \lambda^2 T_{eq}$ equilibrium

Since evolution slows down during turbulent cascade system evolves towards quasi-stationary non-equilibrium solution, describing turbulent energy transport towards larger momenta



Eventually the dissipated turbulence reaches a self-similar state

$$\boxed{\underline{NTFP}: \quad n(k,t) = (Qt)^{\alpha} \, n_S\left(\frac{k}{Q}(Qt)^{\beta}\right)}$$

described by dynamical scaling exponents α, β

and scaling function $n_S(x)$

$$n_S(x) \sim \begin{cases} x^{-S_0} & \text{for } x \lesssim 1 \\ \ll 1 & \text{for } x \gtrsim 1 \end{cases}$$

where $S_0 = m + d$ is the KZ exponent associated with turbulent energy cascade

Since $|V_{k12}|^2 \sim \frac{1}{2\omega_k \omega_1 \omega_2}$ one has $m = -\frac{3}{2}$

for relativistic scalar theory $(z=1)$

Novelty here is scaling in time associated with dynamical exponents α, β

↳ Should be constrained by kinetics governing the process

Conservation laws:

$$E = \int d^dk \; \omega_k \; n(k,t) = \text{const}$$

self-similarity

$$= \int d^dk \; \omega_k \; (Qt)^\alpha \; n_S\!\left(\frac{k}{Q}(Qt)^\beta\right)$$

setting $x = \frac{k}{Q}(Qt)^\beta$ and using $\omega_k \simeq k$

$$= \underbrace{\left(Q^{d+1} \int d^dx \; x \, n_S(x)\right)}_{=\text{const}} (Qt)^{\alpha-(d+1)\beta}$$

Since $E = \text{const}$ ⇒ Scaling relation $\boxed{\alpha = (d+1)\beta}$

Similarly, a second scaling relation can be obtained directly from the kinetic equation, by inserting the scaling ansatz

LHS: $$\frac{\partial}{\partial t} n(k,t) = \underbrace{(Qt)^{\alpha-1}}_{\text{time dependent factor}} \underbrace{\left[\alpha Q \, n_S(x) + \beta Q \, x \frac{d}{dx} n_S(x)\right]}_{=\text{const}}$$

RHS: $I[n](k,t) = \int d^dk_1 d^dk_2 \, \tilde{w}(k \to k_1 k_2)$

$[n_1 n_2 - n_k(n_1+n_2)]$ + other terms

Self-similarity $n(k,t) = (Qt)^\alpha \, n_s\left(\frac{k}{Q}(Qt)^\beta\right)$

defining $x = \frac{k}{Q}(Qt)^\beta$ and similarly for x_1, x_2

$$I[n](k,t) = (Qt)^\omega \, Q^{2d} \int d^dx_1 \, d^dx_2$$

$$\tilde{w}(xQ \to x_1Q \, x_2Q)$$

$$\left[n_s(x_1)\, n_s(x_2) - n_s(x)\left(n_s(x_1) + n_s(x_2)\right)\right]$$

where $\omega = \underbrace{2\alpha}_{n's} - \underbrace{2d\beta}_{\text{integrals}} + \underbrace{(d+1)\beta}_{\substack{\text{energy}\\ \text{momentum}\\ \text{conservation}}} + \underbrace{3\beta}_{|V_{kk_1k_2}|^2}$

So our summary kinetic equation reads

$$(Qt)^{\alpha-1} \underbrace{\left[\alpha Q\, n_s(x) + \beta Q\, x \frac{\partial}{\partial x} n_s(x)\right]}_{= \text{const}} = (Qt)^\omega \underbrace{I[n_s](xQ)}_{= \text{const}}$$

$\Rightarrow$ Decomposes into integro differential equation for scaling function $n_s(x)$ and simple scaling relation matching the time dependence between LHS & RHS

$$\alpha - 1 = \omega$$

Based on the conservation laws & dynamical scaling equation

$$\alpha = (d+1)\beta$$

$$\alpha - 1 = 2\alpha - 2d\beta + (d+1)\beta + 3\beta$$

yielding $\boxed{\beta = -\frac{1}{5} \text{ and } \alpha = -\frac{(d+1)}{5}}$

Dynamical scaling exponents α, β for self-similar scaling are universal in the same sense as KZ exponents so

→ independent of microscopic details of the theory

Note that many important features are readily obtained by dynamical scaling exponents α, β

e.g. we can estimate thermalization time by asking when energy has been transported all the way to momenta

$$p \sim T_{eq} \sim \lambda^{-1/4} Q$$

Since $p_{car} \sim Q\,((Qt)^{-\beta}$ we get

$$p_{car} \overset{!}{\sim} T_{eq} \Rightarrow (Qt_{eq})^{-\beta} \sim \lambda^{-\frac{1}{4}}$$

$$\overset{\beta=-\frac{1}{5}}{\Rightarrow} t_{eq} \sim Q^{-1} \lambda^{-\frac{5}{4}} \sim$$