Recap:

Discussed stationary turbulent solutions of kinetic equations

→ Describe stationary stationary transport of conserved quantity across a large separation of scales

Best example is the Richardson cascade which describes turbulence of surface waves in fluid

Dynamics described by kinetic equation for statistically homogeneous & isotropic system

$$\frac{\partial}{\partial t} \underbrace{n(k,t)}_{\text{occupation number}} = \underbrace{I[n](k,t)}_{\text{wave interaction for weakly non-linear waves}} + \text{source/sink}$$

Depending on the system one typically has 3-wave or 4-wave interactions as dominant process

While standard theory of turbulence is for classical wave systems, same discussion applies to kinetic equation for classical particles

→ see e.g. exercises

We focused on 3-wave interactions $\xrightarrow{k} \langle^{k_1}_{k_2}$

$$I[n](k,t) = \frac{1}{2}\int d^dk_1\, d^dk_2\; \tilde{w}(k\to k_1 k_2)\,\left[n_1 n_2 - n_k(n_1+n_2)\right]$$
$$- \tilde{w}(k_1\to k k_2)\,\left[n_k n_2 - n_1(n_k+n_2)\right]$$
$$- \tilde{w}(k_2\to k_1 k)\,\left[n_k n_1 - n_2(n_k+n_1)\right]$$

where $\tilde{w}(k\to k_1 k_2)$ is transition probability

$$\tilde{w}(k\to k_1 k_2) = (2\pi)\, \delta^{(d)}(k-k_1-k_2)\; \delta(\omega_k-\omega_1-\omega_2)\; |V_{k k_1 k_2}|^2$$

Now the theory of stationary turbulence is best developed for scale invariant systems,

Scale invariance: $\lambda>0$

$$V_{\lambda k \lambda k_1 \lambda k_2} = \lambda^m V_{k k_1 k_2}$$
$$\omega(\lambda k) = \lambda^z\, \omega(k)$$

for which stationary turbulent solutions are of the form that $n(k)$ is power law

$$n(k) = n_0 \left(\frac{k}{k_0}\right)^{-s_0}, \Rightarrow n(\lambda k) = \lambda^{-s_0}\, n(k)$$

Kolmogorov - Zakharov spectrum

In order to determine the scaling exponent s_0 we looked at stationary solutions of kinetic equation.

Stationary equilibrium takes the form

$$\frac{\partial}{\partial t} n(k,t) \stackrel{!}{=} 0$$

such that within inertial range of momenta

$$I[n](k,t) = 0$$

Unlike equilibrium solutions, the stationary turbulent solutions satisfy balance rather than detailed balance

Can be identified using Zakharov transformation

Specifically for a statistically isotropic system, we can express collision integral as

$$I[n](k) = \frac{1}{2\Omega^{(d)}} \int d\Omega_k^d \int d\Omega_{k_1}^d \int d\Omega_{k_2}^d$$

$$\int dk_1\, k_1^{d-1} \int dk_2\, k_2^{d-1}$$

$$\Big[\underset{A}{R(k|k_1 k_2)} - \underset{B}{R(k_1|k k_2)} - \underset{C}{R(k_2|k k_1)} \Big]$$

Idea → to exploit scale invariance of $R(k_1|k k_2)$
to map different terms into the same form

e.g. (B) → (A)

$$\vec{k} = \vec{k}_1 \left|\frac{\vec{k}}{\vec{k}_1}\right| \qquad \vec{k}_1 = \vec{k}\left|\frac{\vec{k}}{\vec{k}_1}\right| \qquad \vec{k}_2 = \vec{k}_2\left|\frac{\vec{k}}{\vec{k}_1}\right|$$

performing all the steps

$$I[n](k) = \frac{1}{2k^d}\int d\Omega_k \int d\Omega_{k_1}\int d\Omega_{k_2}\int dk_1\, k_1^{d-1}\int dk_2\, k_2^{d-1}$$
$$R(\vec{k}|\vec{k}_1\vec{k}_2)\left[1 - \left|\frac{\vec{k}}{\vec{k}_1}\right|^{\xi} - \left|\frac{\vec{k}}{\vec{k}_2}\right|^{\xi}\right]$$

where $\xi = 2d - z + 2m - 2s_0$

Now one sees that the integrand vanishes
whenever $[\,\cdot\,]$ is proportional to a
collisional invariant

Since energy is the only relevant conserved
quantity for 3-wave scattering, so requiring

$$\left[1 - \left|\frac{\vec{k}}{\vec{k}_1}\right|^{\xi} - \left|\frac{\vec{k}}{\vec{k}_2}\right|^{\xi}\right] \stackrel{!}{=} c\left[\omega(k) - \omega(k_1) - \omega(k_2)\right]$$
$$= c\,\omega(k)\left[1 - \left|\frac{k_1}{k}\right|^{z} - \left|\frac{k_2}{k}\right|^{z}\right]$$

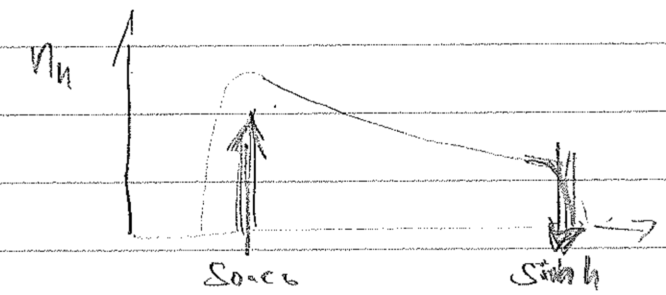
$$\Rightarrow c = \frac{1}{\omega(k)} \qquad \boxed{\xi = -z}$$

We conclude that for $\boxed{s_0 = m + d}$ we have a stationary solution

Balance equations in k-space & stationary transport

Will focus on case of 3-wave interactions
where interactions can change the overall
number density, since net-momentum also
vanishes for statistically isotropic systems
→ energy is only relevant conserved
quantity in the system

Now as a function of the wave-number k
we can consider the typical case of
a <u>direct cascade</u>



globally $$\frac{dE}{dt} = \dot{E}_{in} - \dot{E}_{out}$$

So to reach a stationary state we need
$\dot{E}_{in} = \dot{E}_{out}$. In this case energy is injected into the system
at some characteristic scale k_{source}
transported all the way to k_{sink}
where it is removed from the system

Now if we look differentially in wave number

$$\mathcal{E}(k,t) \equiv \omega(k)\, n(k,t) \quad \text{such that} \quad \int d^dk\, \mathcal{E}(k,t) = E$$

now we have

$$\frac{\partial}{\partial t} \mathcal{E}(k,t) = \omega(k)\, I[n](k,t) + \dot{E}_{in}(k) - \dot{E}_{out}(k)$$

which can be recast into the form of a continuity equation in momentum space by identifying

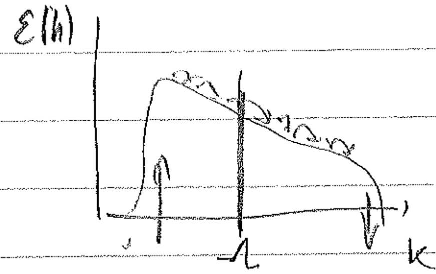
$$\omega(k)\, I[n] = -\vec{\nabla}_k \vec{J}_{\mathcal{E}}(k,t)$$

So in the inertial range of momenta far away from source and sink $(\dot{E}_{in}(k) \approx \dot{E}_{out}(k) \approx 0)$

$$\frac{\partial}{\partial t} \mathcal{E}(k,t) + \vec{\nabla}_k \vec{J}_{\mathcal{E}}(k,t) = 0$$

we have a continuity equation for energy transport in wave number space.
(think again of Richardson cascade)

Within inertial range of wave numbers energy flux should be independent of wave number, to avoid accumulation of energy at intermediate scales



$$J(k,t) = \underbrace{\int_{\partial\Lambda}}_{\text{momentum shell}} \vec{J}_\mathcal{E}(k,t)\, d\vec{\sigma}_k = \int^{\Lambda} \nabla_k \vec{J}_\mathcal{E}(k,t)\, dk = -\int^{\Lambda} d^d k\ \omega(k)\, I[n](k,t)$$

So the condition that the flux is scale invariant becomes

$$J(k,t) = -\int^{\Lambda} d^d k\ \omega(k)\, I[n](k)$$

$$= -\int^{\Lambda} d^d k\ \lambda^{-z} \omega(\lambda k)\ \lambda^{-\mu} I[n](\lambda k)$$

so choosing $\lambda = \frac{k_0}{k}$

$$= -\int^{\Lambda} d\Omega_k^d\, k^{d-1} dk\ k^{\mu+z}\ k_0^{-(\mu+z)} \omega(k_0)\, I[n](k_0)$$

$$= -\Omega^d\ \frac{\Lambda^{d+\mu+z}}{d+\mu+z}\ k_0^{-(\mu+z)} \omega(k_0)\, I[n](k_0)$$

(up to log corrections)

We deduce that flux is scale independent when

$$d + \mu + z = 0 \qquad S_0 = m + d$$

Spectrum $\boxed{n(k) = k^{-(m+d)}}$ is associated with constant energy flux

→ stationary solution with finite flux

Note that scaling exponent $S_0 = m + d$ insensitive to microscopics → universal

but sensitive to scaling properties of interaction matrix element (m) and dimensionality (d) of the system