Stationary non-equilibrium solutions

Books: Kolmogorov Spectra of turbulence, Wave turbulence, Zakharov, L'vov, Falkovich

Based on positivity of entropy production

generally expect system to approach some kind of equilibrium state at asymptotically late times

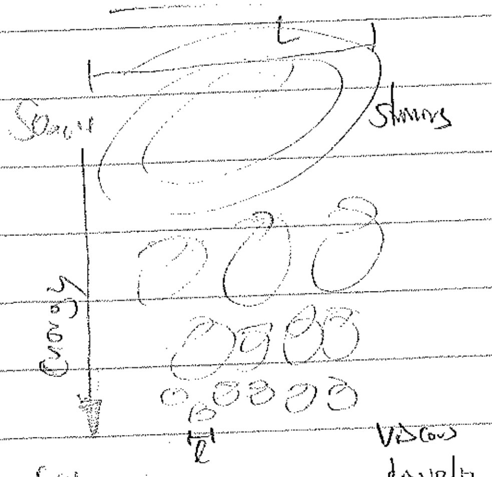
→ so far only studied many body systems in isolation, i.e. no external influences for example due to driving forces

Situation can change quite dramatically when considering open systems, where stationarity requires a subtle balance between driving and dissipative forces

Consider for example turbulence of surface waves in a fluid

→ energy injected at stirring scale L source creating large eddies

Richardson cascade



→ fragmentation into smaller and smaller eddies

→ very small eddies of size ℓ small subject to viscous dissipation

While such a system is highly dynamical, continuously transporting energy from large scales to small scales its statistical features such as e.g. typical wave amplitude as a function of wave number

Question: Can we make a statement about the statistical properties of the stationary state

Generally this is complicated because different stirring or different dissipation mechanisms will lead to different stationary solutions

However it turns out that when there is a large separation of scales between source and sink the dynamics on scales

$$\ell_{source} \ll \ell \ll \ell_{sink}$$

is often insensitive to the details and exhibits universal properties across different physical systems

Describes state of stationary turbulence:

Stationary: statistical properties are time independent

turbulence: associated with transport of conserved quantity across large separation of scales

Concepts of turbulence can also a role in determining non-equilibrium behaviour of isolated systems (i.e. no driving/dissipation)
→ will be briefly discussed later

(Weak) Wave turbulence

Generally good to have example of Richardson cascade in mind, but theory of wave turbulence more general

→ many applications plasma physics, non-linear optics, ...

see book for details

Generally we associate with a wave a perturbation of a continuous medium propagating with a certain phase velocity

If the medium is weakly non-linear different waves are independent of each other up to small non-linear corrections

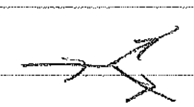
$$\underbrace{\phi(t,\vec r)}_{\substack{\text{physical field}\\ \text{e.g. } \rho, \vec v, \dots}} = \int \frac{d^dk}{(2\pi)^d}\left[\underbrace{a(k,t)}_{\substack{\text{wave amplitude}\\ \text{of perturbation}\\ \text{with definite } k}} e^{i\vec k\vec r}\,\phi_k + a^*(k,t)\, e^{-i\vec k\vec r}\,\phi_k^*\right]$$

where to linear order

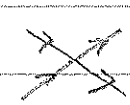
$$\frac{\partial}{\partial t} a(k,t) = -i\omega\, a(k,t) + \text{non-linear corrections}$$

where the non-linear corrections can include

e.g.



or



If non-linear corrections are small such that typically waves travel a long distance between interactions, dynamics can be approximated as scattering of uncorrelated waves (RPA)

→ Kinetic equation for occupation number

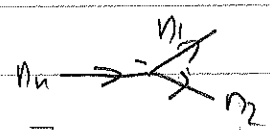
$$n(k,t) = \frac{1}{V}\langle a(k,t)\, a^*(k,t)\rangle$$

!! for statistically homogeneous system

$$\frac{\partial n(k,t)}{\partial t} = I[n](k,t)$$

Depending on the system under consideration, one typically has either three wave interactions or four wave interactions as dominant process

3-wave:



$$I[n](k,t) = \frac{1}{2}\int d^dk_1\, d^dk_2 \Big\{ \tilde{W}(k\to k_1 k_2)\left[n_1 n_2 - n_k(n_1+n_2)\right]$$

$$+ 2\,\tilde{W}(k_1\to k k_2)\left[n_k n_2 - n_1(n_k+n_2)\right]\Big\}$$

where $\tilde{W}(k\to k_1 k_2) = (2\pi)\,\delta^{(d)}(k-k_1-k_2)\,\delta(\omega(k)-\omega(k_1)-\omega(k_2))\,|V_{kk_1k_2}|^2$

Note decay process goes as $n_k(n_1+n_2)$

⇝ classical field behaviour

Now whether such three wave interactions are allowed crucially depends on the dispersion relation, i.e. whether

momentum conservation $\vec{k} = \vec{k}_1 + \vec{k}_2$

& energy conservation $\omega(\vec{k}_1 + \vec{k}_2) = \omega(\vec{k}_1) + \omega(\vec{k}_2)$

can be satisfied simultaneously

Now for a typical dispersion

$$\omega(\vec{k}) = c|\vec{k}| + \varepsilon|\vec{k}|^2 + \dots$$

one finds that this is only possible if $\varepsilon \geq 0$ where for $\varepsilon = 0$ $\vec{k}_1 // \vec{k}_2$ have to be exactly parallel

Intuitively the magnitude of $\vec{k}_1 + \vec{k}_2$ will be less than $|\vec{k}_1| + |\vec{k}_2|$ and therefore energy $\omega(|\vec{k}|)$ has to grow faster than linear to open up the phase space

If these wave interactions are forbidden the dynamics is typically dominated by the 4 oth order process

4 wave:

$$I[n](k,t) = \frac{1}{2}\int d^dk_1 d^dk_2 d^dk_3 \; \tilde{w}(k k_1 \to k_2 k_3)$$

$$\left[n_2 n_3 (n_1 + n_k) - n_1 n_k (n_2 + n_3) \right]$$

with $\tilde{w}(k k_1 \to k_2 k_3) = (2\pi)\, \delta(\omega_k + \omega_1 - \omega_2 - \omega_3)$

$$\delta^{(d)}(k + k_1 - k_2 - k_3)$$

$$\frac{|T_{k123}|^2}{2}$$

Source / Sink

Now ~~if~~ we add an explicit source term to account for presence of source and sink

Whole equation gets modified

$$\frac{\partial}{\partial t} n(k,t) = I[n](k,t) + \underbrace{\Gamma(k)}_{\text{forcing}} n(k,t)$$

$\Gamma(k) > 0$ source $\Gamma(k) < 0$ damping

Now in the inertial range of momenta we have $\Gamma(k) \simeq 0$ ($k_{source} \ll k \ll k_{sink}$)

So stationary solutions need to satisfy

$$\frac{\partial}{\partial t} n(k,t) = 0 \Rightarrow \boxed{I[n](k,t) = 0}$$

stationarity condition insensitive to forcing characteristics (as long as interactions are sufficiently local in k-space)

Stationary solutions

Can search for stationary solutions as before, considering detailed balance and balance solutions

We have for detailed balance that effects of each process are cancelled by its inverse processes

3 wave interaction $\Rightarrow$ $n_1 n_2 = n_k (n_1 + n_2)$

Now we can further manipulate this expression

$$n_1 n_2 n_4 \left(\frac{1}{n_1} + \frac{1}{n_2} - \frac{1}{n_4} \right) = 0$$

cancellation requires that

$\Rightarrow$ $\frac{1}{n_1} + \frac{1}{n_2} - \frac{1}{n_4}$ is proportional to conserved quantity

Stationary equilibrium solutions

$$n_k = n(k,t) = \frac{T}{\omega(k) - \vec{k}\vec{V}} \quad \text{3 wave}$$

$$n(k,t) = \frac{T}{\omega(k) - \vec{k}\vec{V} - \mu} \quad \text{4 wave}$$

Rayleigh-Jeans distribution for classical waves

$\rightarrow$ analogous to Maxwell-Boltzmann distribution for classical particles

Now besides the equilibrium solutions solutions

detailed balance there are also other stationary solutions satisfying

balance but this depends on properties of collision integral

In many physically relevant situations, the system is at least approximately scale invariant meaning that if we look at collision integral

$$I[n](k,t) = \frac{1}{2}\int d^dk_1 \int d^dk_2 \; \tilde{W}(k \to k_1 k_2)\left[n_1 n_2 - n_k(n_1+n_2)\right]$$
$$+ 2\tilde{W}(k_1 \to k k_2)\left[n_k n_2 - n_1(n_k+n_2)\right]$$

$$\tilde{W}(k \to k_1 k_2) = (2\pi)\,\delta^{(d)}(k-k_1-k_2)\,\delta(\omega(k)-\omega(k_1)-\omega(k_2))\,|V_{k k_1 k_2}|^2$$

and consider a rescaling of all momenta by a factor $\lambda > 0$ we have

$$V_{\lambda k, \lambda k_1, \lambda k_2} = \lambda^m V_{k k_1 k_2}$$
$$\omega(\lambda k) = \lambda^z \omega(k) \qquad \text{(meaning that } \omega(k) \text{ is a power law)}$$

then it is natural to also search for power law solutions of $n(k,t)$ such that in the stationary case

$$n(\lambda k) = \lambda^{-s_0} n(k)$$

One then finds that the entire collision ~~integral exhibits a scaling~~

$$I[n](\lambda k, t) = \lambda^{\mu} I[n](k,t)$$

where rescaling $k_{1/2} \lambda \hat{k}_{1/2}$

$$\mu = \underbrace{2d}_{} - \underbrace{d - z}_{s'_0} + 2m - 2s_0$$

Can now look at stationary solutions of kinetic equations for scale invariant systems

We will focus on 3-wave interaction (see e.g. book for 4-wave case)

Stationary condition: $I[n](k) = 0$ within inertial range of momenta k

$$I[n](k) = \frac{1}{2}\int d^dk_1\, d^dk_2 \Big[R(k,k_1,k_2) - R(k_1,k,k_2) - R(k_2,k,k_1) \Big]$$

where $R(k,k_1,k_2) = \tilde{w}(k \to k_1,k_2)\big(n_1 n_2 - n_k(n_1+n_2)\big)$

If we are interested in statistically isotropic systems, can integrate over angle $\int d\Omega_k$ without loss of generality (alternatively could perform angular integrations in collision integral as done in book)

$$I[n](k) = \frac{1}{2\Omega_d}\int d\Omega^d_k \int d\Omega^d_{k_1} \int d\Omega^d_{k_2} \int dk_1\, k_1^{d-1} \int dk_2\, k_2^{d-1}$$

$$\Big[\underbrace{R(\vec{k},\vec{k}_1,\vec{k}_2)}_{A} - \underbrace{R(\vec{k}_1,\vec{k},\vec{k}_2)}_{B} - \underbrace{R(\vec{k}_2,\vec{k},\vec{k}_1)}_{C} \Big]$$

Idea is now to perform <u>Zakharov transformations</u> which is a transformation that maps different terms into same form up to scale factor

Consider (B) which can be mapped to (A) by the following transformation

$$\vec{k}_1 = \tilde{\vec{k}} \left|\frac{k}{\tilde{k}_1}\right| \qquad \vec{k}_2 = \tilde{\vec{k}}_2 \left|\frac{k}{\tilde{k}_1}\right| \qquad \underbrace{\vec{k} = \tilde{\vec{k}}_1 \left|\frac{k}{\tilde{k}_1}\right|}_{\text{same magnitude as } \vec{k} \text{ but different directions}}$$

then $R(\vec{k}_1, \vec{k}_1, \vec{k}_2)$ gets mapped to

$$R\left(\tilde{\vec{k}} \left|\frac{k}{\tilde{k}_1}\right|, \tilde{\vec{k}}_1 \left|\frac{k}{\tilde{k}_1}\right|, \tilde{\vec{k}}_2 \left|\frac{k}{\tilde{k}_1}\right|\right)$$

$$= R(\lambda \tilde{\vec{k}}, \lambda \tilde{\vec{k}}_1, \lambda \tilde{\vec{k}}_2) \quad \text{with } \lambda = \left|\frac{k}{\tilde{k}_1}\right|$$

$$\overset{\text{scale invariance}}{=} \lambda^{m-2d} R(\tilde{\vec{k}}, \tilde{\vec{k}}_1, \tilde{\vec{k}}_2)$$

Note that transformation requires separate convergence of integrals → locality of interactions

Now also need to do this for integrals over k

$$\int_0^\infty dk_1\, k_1^{d-1} = \int_0^\infty d\left(k\left|\frac{k}{\tilde{k}_1}\right|\right) \left(k\left|\frac{k}{\tilde{k}_1}\right|\right)^{d-1}$$

$$= \int_0^\infty d\left(\frac{1}{\tilde{k}_1}\right) \frac{k^{2d}}{\tilde{k}_1^{d-1}}$$

$$= -\int_\infty^0 d\tilde{k}_1 \frac{1}{\tilde{k}_1^2} \frac{k^{2d}}{\tilde{k}^{d-1}}$$

$$= +\int_0^\infty d\tilde{k}_1\, \tilde{k}_1^{d-1} \left(\frac{k}{\tilde{k}_1}\right)^{2d}$$

$$\int_0^\infty dk_2\, k_2^{d-1} = \int_0^\infty d\tilde{k}_2\, \tilde{k}_2^{d-1} \left|\frac{k}{\tilde{k}_1}\right|^{d}$$

So we get

$$(B) \quad \left(\frac{k}{\tilde{k}_1}\right)^{\mu-2d+2d+d} \quad R(\vec{\tilde{k}}_1, \vec{\tilde{k}}_2, \vec{k}_2)$$

Similarly can map (C) to (A) by interchange role of $\tilde{k}_1$ and $\tilde{k}_2$

$$(C) \quad \left(\frac{k}{\tilde{k}_2}\right)^{\mu+d} \quad R(\vec{\tilde{k}}_1, \vec{k}_1, \vec{\tilde{k}}_2)$$

$$I[n,\tilde{I}](k) = \frac{1}{2\Omega(d)} \int d\Omega_k \int d\Omega_{k_1} \int d\Omega_{k_2} \int dk_1\, k_1^{d-1} \int dk_2\, k_2^{d-1}$$
$$R(\vec{k}, \vec{k}_1, \vec{k}_2) \left[1 - \left(\frac{k}{k_1}\right)^{\mu+d} - \left(\frac{k}{k_2}\right)^{\mu+d} \right]$$

Now we know that this integral vanishes whenever $[\;\;]$ is proportional to conserved quantity

$\Rightarrow$ Can determine scaling exponent

Indeed for $\mu + d = -z$ we have

$$\left[1 - \left(\frac{k}{k_1}\right)^{-z} - \left(\frac{k}{k_2}\right)^{-z} \right] = \frac{1}{k^z} \left[k^z - k_1^z - k_2^z \right]$$

$$\overset{\omega(k) = \omega_0 k^z}{=} \frac{1}{\omega_0 k^z} \left[\omega(k) - \omega(k_1) - \omega(k_2) \right]$$

such that the integral vanishes for all allowed configurations of momenta

$$\mu + d = -z \iff$$

$$2\lambda - d - \cancel{z} + 2m - 2s_0 = -d - \cancel{z}$$

We obtain $\boxed{s_0 = m + d}$ such that

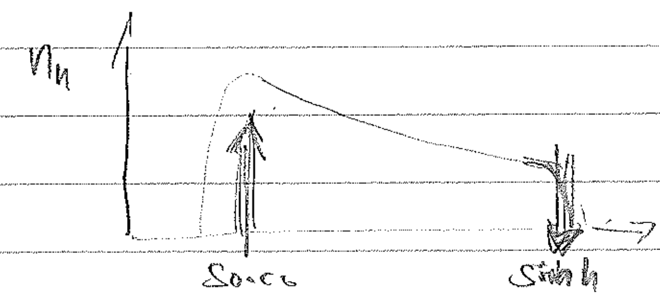
$$n(k) = n_0 \left(\frac{k}{k_0}\right)^{-(m+d)}$$

is a stationary particle solution

Balance equations in k-space & stationary transport

Will focus on case of 3-wave interactions where interactions can change the overall number density, since not-manifolds also vanishes for statistically isotropic system
→ energy is only relevant conserved quantity in the system

Now as a function of the wave-number k we can consider the typical case of a direct cascade



globally $\dfrac{dE}{dt} = \dot{E}_{in} - \dot{E}_{out}$

So to reach a stationary state we need $\dot{E}_{in} = \dot{E}_{out}$. In this case energy is injected into the system at some characteristic scale k_{source} transported all the way to k_{sink} where it is removed from the system

Now if we look differentially on wave number

$$\mathcal{E}(k,t) \equiv \omega(k)\, n(k,t) \qquad \text{such that} \qquad \int d^dk\, \mathcal{E}(k,t) = E$$

now we have

$$\frac{\partial}{\partial t} \mathcal{E}(k,t) = \omega(k)\, I[n](k,t) + \dot{E}_{in}(k) - \dot{E}_{out}(k)$$

which can be recast into the form of a continuity equation in momentum space by identifying

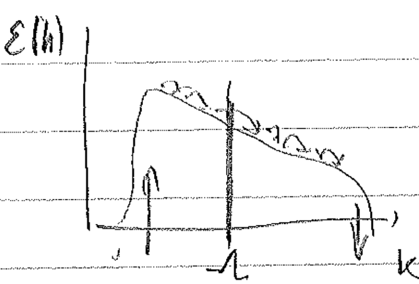
$$\omega(k)\, I[n] = -\vec{\nabla}_k \vec{J}_{\mathcal{E}}(k,t)$$

So in the inertial range of momenta far away from source and sink $\left(\dot{E}_{in}(k) = \dot{E}_{out}(k) = 0\right)$

$$\frac{\partial}{\partial t} \mathcal{E}(k,t) + \vec{\nabla}_k \vec{J}_{\mathcal{E}}(k,t) = 0$$

we have a continuity equation for energy transport in wave-number space.
(think again of Richardson cascade)

Within inertial range of wave numbers energy flux should be independent of wave number, to avoid accumulation of energy at intermediate scales



$$J(k,t) = \underbrace{\int_{\partial\Lambda}}_{\text{momentum shell}} \vec{J}_{\mathcal{E}}(k,t)\, d\vec{\sigma}_k = \int^{\Lambda} \nabla_k \vec{J}_{\mathcal{E}}(k,t)\, d^dk = -\int^{\Lambda} d^dk\ \omega(k)\, I[n](k,t)$$

So the condition that the flux is scale invariant becomes

$$J(k,t) = -\int^{\Lambda} d^dk\ \omega(k)\, I[n](k)$$

$$= -\int^{\Lambda} d^dk\ \lambda^{-z}\omega(\lambda k)\ \lambda^{-\mu}\, I[n](\lambda k)$$

so choosing $\lambda = \frac{k_0}{k}$

$$= -\int^{\Lambda} d\Omega_k^d\, k^{d-1}dk\ k^{\mu+z}\ k_0^{-(\mu+z)}\omega(k_0)\, I[n](k_0)$$

$$= -\Omega^d\, \frac{\Lambda^{d+\mu+z}}{d+\mu+z}\ k_0^{-(\mu+z)}\omega(k_0)\, I[n](k_0)$$

(up to log corrections)

We deduce that flux is scale independent
when

$$d + \mu + z = 0 \qquad s_0 = m + d$$

Spectrum $\boxed{n(k) = k^{-(m+d)}}$ is associated
with constant energy flux
→ stationary solution with finite flux

Note that scaling exponent $s_0 = m + d$ insensitive
to microscopics → universal
but sensitive to scaling properties of interaction
matrix element (m) and dimensionality (d)
of the system