Recap I

Hydrodynamics from Boltzmann eqn

Hydro
- conservation laws
- constitutive relations
- equation of state

Derived balance equations from Boltzmann

continuity equation
$$\frac{\partial}{\partial t}\rho(t,\vec{r}) + \vec{\nabla}_r\left(\rho(t,\vec{r})\,\vec{v}(t,\vec{r})\right) = 0$$

momentum balance
$$\frac{\partial}{\partial t}\rho(t,\vec{r})v^i(t,\vec{r}) + \frac{\partial}{\partial r_j}\left[\rho(t,\vec{r})\,v^i(t,\vec{r})v^j(t,\vec{r}) + \Pi^{ij}(t,\vec{r})\right]$$
$$= \frac{\rho(t,\vec{r})}{m}F^i(t,\vec{r})$$

energy balance
$$\frac{\partial e(t,\vec{r})}{\partial t} + \vec{\nabla}\left(\vec{J}_u(t,\vec{r}) + e(t,\vec{r})\,\vec{v}(t,\vec{r})\right) = -\Pi_{ij}\frac{\partial v^i}{\partial r_j}(t,\vec{r})$$

along with explicit expressions for

stress-tensor $\Pi^{ij}(t,\vec{r}) = \rho(t,\vec{r})\left\langle\left(\frac{p_i}{m} - v_i(t,\vec{r})\right)\left(\frac{p_j}{m} - v_j(t,\vec{r})\right)\right\rangle_{\vec{p}}$

energy flux in lrf $\vec{J}_u(t,\vec{r}) = \frac{1}{2}\rho(t,\vec{r})\left\langle\left(\frac{\vec{p}}{m} - \vec{v}(t,\vec{r})\right)^2\left(\frac{\vec{p}}{m} - \vec{v}(t,\vec{r})\right)\right\rangle_{\vec{p}}$

where
$$\langle\chi\rangle_{\vec{p}} = \frac{1}{n(t,\vec{r})}\int_{\vec{p}}\chi(t,\vec{r},\vec{p})\,f(t,\vec{r},\vec{p})$$

Now in phenomenological treatment in Chapter I the fluxes $\underline{\underline{\Pi}}$ and $\vec{J}_u$ where fixed by linear constitutive relations, i.e. proportional to gradients

Now this concludes part 1) of the derivation of hydrodynamics, what remains to do soon is how constitutive relations emerge from Boltzmann framework

Generally this will result in reduction of dof's to local thermodynamic properties and e.g. their gradients
→ Necessary to truncate Boltzmann equation

Note that different truncations are possible corresponding to different expansion schemes / power countings

→ active topic of research in particular for relativistic theories

We will discuss traditional way based on Chapman-Enskog expansion

which can be viewed as a short expansion in Knudsen number

Chapman Enskog expansion

Since Kn does not appear explicitly in Boltzmann equation, consider instead $\varepsilon \ll 1$

$$\left(\frac{\partial}{\partial t} + \frac{\vec{p}}{m}\partial_r + \vec{F}\partial_p\right) f = \frac{1}{\varepsilon} C[f]$$

and expands

$$f(t,\vec{r},\vec{p}) = f^{(0)}(t,\vec{r},\vec{p}) + \varepsilon f^{(1)}(t,\vec{r},\vec{p}) + \dots$$

under the assumption that :

$$f(t,\vec{r},\vec{p}) = f\left(T(t,\vec{r}), n(t,\vec{r}), \vec{v}(t,\vec{r}), \vec{p}\right)$$

such that derivatives w.r.t. coordinates can be expressed in terms of derivatives of thermodynamic variables, which are obtained via matching conditions

$$n(t,\vec{r}) = \int_{\vec{p}} f(t,\vec{r},\vec{p})$$

$$\vec{v}(t,\vec{r}) = \int_{\vec{p}} \frac{\vec{p}}{m} f(t,\vec{r},\vec{p})$$

$$\frac{3}{2} n(t,\vec{r})\, k_B T(t,\vec{r}) = \int_{\vec{p}} \frac{(\vec{p}-m\vec{v}(t,\vec{r}))^2}{2m} f(t,\vec{r},\vec{p})$$

Note that expansion may not converge (e.g. asymptotic series)

Now in order to develop an expansion where perturbations $f^{(1)}, f^{(2)}, \ldots$ are in the subspace orthogonal to the conserved quantities, we want to absorb the full information about $n, \vec{v}, T$ also to leading order distribution

$$n(t,\vec{r}) = \int_{\vec{p}} f(t,\vec{p},\vec{r}) = \int_{\vec{p}} f^{(0)}(t,\vec{p},\vec{r})$$

and similarly for other quantities

Zeroth order : Ideal fluid dynamics

$\mathcal{O}\left(\frac{1}{\varepsilon}\right) \quad C\left[f^{(0)}\right] = 0$

$\Rightarrow f^{(0)}$ is local equilibrium solution

$$f^{(0)} = n(\vec{r},t) \left(\frac{2\pi\hbar^2}{mk_BT(t,\vec{r})}\right)^{3/2} \exp\left(\frac{-(\vec{p}-m\vec{v}(t,\vec{r}))^2}{2mk_BT(t,\vec{r})}\right)$$

where $n, v(t,\vec{r})$ is the same as $n, v(t,\vec{r})$ in continuity equation, as guaranteed by orthogonality condition

Based on explicit form of $f^{(0)}$ can now calculate $\vec{J}_u^{(0)}, \Pi^{ij\,(0)}$ at zeroth order

$$\Pi^{ij(0)}(t,\vec{r}) = n(t,\vec{r}) \left(\frac{2\pi\hbar^2}{mk_BT(t,\vec{r})}\right)^{3/2} \int_{\vec{p}} \frac{(p-mv(t,\vec{r}))^i (p-mv(t,\vec{r}))^j}{m} \exp(\;)$$

so performing angular integration

$$\Pi^{ij(0)}(t,\vec{r}) = \frac{2}{3}\varepsilon(t,\vec{r})\,\delta^{ij} = n(t,\vec{r})\,k_BT(t,\vec{r})\,\delta^{ij}$$

so

$$\boxed{\Pi^{ij\,(0)}(t,\vec{r}) = P(t,\vec{r})\,\delta^{ij}}$$

conversely for

$$\vec{J}_u^{(0)}(t,\vec{r}) = n(t,\vec{r}) \left(\frac{2\pi\hbar^2}{mk_BT(t,\vec{r})}\right)^{3/2} \int_{\vec{p}} \frac{(\vec{p}-m\vec{v}(t,\vec{r}))^2}{2m} \left(\frac{\vec{p}}{m}-\vec{v}(t,\vec{r})\right) \exp(\dots)$$

vanishes since integrand is odd function of $\vec{p}-m\vec{v}(t,\vec{r})$

$$\boxed{\vec{J}_u^{(0)}(t,\vec{r}) = 0}$$

So using this results we get

$$\rho\left[\frac{\partial\vec{v}}{\partial t}+\left(\vec{v}\cdot\vec{\nabla}\right)\vec{v}\right]=-\vec{\nabla}P+\frac{\rho}{m}\vec{F}$$

$$\frac{\partial\varepsilon}{\partial t}+\vec{\nabla}\left(e\vec{v}\right)=-P\left(\vec{\nabla}\vec{v}\right)$$

which are EOMs of ideal fluid dynamics

First order:

$$\mathcal{O}(\varepsilon^0) \quad \left(\frac{\partial}{\partial t} + v\vec{\nabla}_r + \vec{f}\,\vec{\nabla}_p\right) f^{(0)} = SC[f^{(0)}, f^{(1)}]$$

Will focus on RTA: $SC[f^{(0)}, f^{(1)}] = -\frac{1}{\tau_R} f^{(1)}$

and hence

$$f^{(1)} = -\tau_R \left(\frac{\partial}{\partial t} + v\vec{\nabla}_r + \vec{f}\,\vec{\nabla}_p\right) f^{(0)}$$

now the challenge to a certain extent is to evaluate the various derivatives, e.g.

$$\frac{\partial}{\partial t} f^{(0)} = \frac{\partial n}{\partial t}\frac{\partial f^{(0)}}{\partial n} + \frac{\partial T}{\partial t}\frac{\partial f^{(0)}}{\partial T} + \frac{\partial \vec{v}}{\partial t}\frac{\partial f^{0}}{\partial \vec{v}}$$

where to this order in the expansion it is consistent to use ideal hydrodynamic equations for $n, \vec{v}, T$ to express e.g.

$$\frac{\partial n}{\partial t} = -\vec{\nabla}(n\vec{v})$$

and similarly for other terms

Derivatives ~~on~~ of $\frac{\partial}{\partial t}\vec{v}, \frac{\partial}{\partial t}T$ are of higher order and accounted for in higher order of expansion

Denoting $\vec{u}_{\vec{p}} = \frac{\vec{p}}{m} - \vec{V}(t,\vec{r})$ one finds
(see Chapter 7.5 in Pathria for details)

$$f^{(1)} = -\tau_R f^{(0)} \left[\frac{1}{T} \left(\vec{u}_{\vec{p}} \cdot \vec{\nabla} T\right) \left(\frac{m}{2k_BT} \vec{u}_{\vec{p}}^{\,2} - \frac{5}{2} \right) + \frac{m}{2k_BT} \left(\frac{\partial V_k}{\partial x_l} + \frac{\partial V_l}{\partial x_k} \right) \left(u_{\vec{p}}^k u_{\vec{p}}^l - \frac{1}{3} \delta^{kl} u_{\vec{p}}^2 \right) \right]$$

and we can now proceed to calculate
first order corrections to energy flux J_u
and stress tensor Π^{ij}

$$\Pi^{ij} = \Pi^{ij(0)} + \Pi^{ij(1)}$$

$$\Pi^{ij(1)} = m \int_{\vec{p}} u^i u^j f^{(1)}$$

since first term is odd in u

$$= -\frac{m^2 \tau_R}{2k_BT} \left(\frac{\partial V_k}{\partial x_l} + \frac{\partial V_l}{\partial x_k} \right) \int_{\vec{p}} u_i u_j \left(u^k u^l - \frac{1}{3} \vec{u}^2 \delta^{kl} \right) f^{(0)}$$

Decompose integral in basis of symmetric tensors

$$I^{ijkl} = \int_p u_i u_j \left(u^k u^l - \frac{1}{3} u^2 \delta^{kl}\right) f^{(0)} = A\,\delta_{ij}\delta_{kl} + B\left(\frac{\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}}{2}\right)$$

$$\delta^{ij}\delta^{kl} I^{ijkl} = 0 \qquad \delta^{kl}\left(u^k u^l - \frac{1}{3} u^2 \delta^{kl}\right) = 0$$

$$9A + 3B = 0$$

$$A = -\frac{1}{3} B$$

$$\delta^{ik}\delta^{jl} I^{ijkl} = 3A + B\left(\frac{9+3}{2}\right)$$
$$= -B + 6B = 5B$$

$$B = \frac{1}{5}\delta^{ik}\delta^{jl}\int_{\vec p} u_i u_j \left(u^k u^l - \frac{1}{3}\vec u^2 \delta^{kl}\right) f^{(0)}$$
$$= \frac{1}{5}\int_{\vec p} \vec u^4 \left(1 - \frac{1}{3}\right) f^{(0)}$$
$$= \frac{2}{15}\underbrace{\int_{\vec p} \vec u^4 f^{(0)}}_{= 15\, n \frac{(k_B T)^2}{m^2}} = 2n\frac{(k_B T)^2}{m^2}$$

So collecting everything we find

$$\Pi^{ij(1)} = -\frac{m^2 \tau_R}{2 k_B T}\, 2n \frac{(k_B T)^2}{m^2}\left(\frac{\partial v_k}{\partial x_l} + \frac{\partial v_l}{\partial x_k}\right) \times \left(-\frac{1}{3}\delta_{ij}\delta_{kl} + \frac{\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}}{2}\right)$$

$$= -n k_B T \tau_R \underbrace{\left(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i} - \frac{2}{3}\delta^{ij}\vec\nabla\vec v\right)}_{= 2\sigma^{ij}}$$

So $\boxed{\Pi^{ij(1)} = -2\eta\, \sigma^{ij}}$

with $\eta = n k_B T \tau_R$

Can also compute dissipative corrections
to internal energy flux

$$\vec{J}_u = \vec{J}_u^{(0)} + \vec{J}_u^{(1)}$$

we found from linear constitutive relations
and symmetries that

$$\vec{J}_u = -\mathcal{K} \vec{\nabla} T$$

now at leading order $\vec{J}_u^{(0)} = 0$
so only contribution

$$\vec{J}_u = \int_{\vec{p}} \frac{(m\vec{u}_p)^2}{2m} \vec{u}_p f^{(1)}$$

$$J_u^i = -\underbrace{\int_{\vec{p}} \frac{\tau_R}{T} \frac{(m\vec{u}_p)^2}{2m} \vec{u}_p^i \vec{u}_p^j \left(\frac{m}{2k_BT}\vec{u}_p^2 - \frac{5}{2}\right) f^{(0)}}_{} \frac{\partial T}{\partial r_j}$$

so upon performing angular integration $= \mathcal{K}\,\delta^{ij}$

$$\mathcal{K} = \frac{1}{3}\int_{\vec{p}} \frac{\tau_R}{T} \frac{u_p^2}{2m} \vec{u}_p^2 \left(\frac{m}{2k_BT}\vec{u}_p^2 - \frac{5}{2}\right) f^{(0)}$$

$$= \frac{\tau_R}{3Tm^2} \int_{\vec{p}} \frac{(m\vec{u}_p)^2}{2m} (m\vec{u}_p)^2 \left(\frac{(m\vec{u}_p)^2}{2mk_BT} - \frac{5}{2}\right) f^{(0)}$$

$$= \frac{\tau_R}{3Tm^2} \underbrace{e}_{=\frac{3}{2}nk_BT} \underbrace{\left(\frac{35}{2} - \frac{25}{2}\right)}_{=5} m k_B T$$

$$\boxed{\mathcal{K} = \frac{5}{2} \frac{n k_B^2 T \tau_R}{m}}$$

We see that transport coefficients η, κ are proportional to relaxation time τ_R which generally should be a function of $T(t,\vec{r}), n(t,\vec{r})$

→ Smaller relaxation time → smaller avg. deviation from local equilibrium

→ Smaller viscosity → more ideal fluid

Can estimate further that

$$\tau_R \sim \tau_{mfp} \sim \frac{1}{n \sigma_{tot} \langle v \rangle} \qquad \langle v \rangle \sim \sqrt{\frac{k_B T}{m}}$$

$$\Rightarrow \eta = n k_B T \tau_R \sim \frac{\sqrt{m k_B T}}{\sigma_{tot}}$$

approximately independent of density n and inversely proportional to total cross section σ_{tot}