Recap: Calculation of transport coefficients

Discussed in more detail the case of electric conduction and corrected mistake in previous discussion.

By considering balance equation for momentum for multicomponent systems

$$\sum_i \langle m_i n_i \vec{v}_i \rangle = \sum_i q_i n_i \vec{E}$$

System is subject to constant acceleration unless it is electrically neutral

$$\sum_i q_i n_i = 0$$

Can search for stationary solutions for electrically neutral system

→ calculation of transport coefficients applies as discussed

Developed a picture of conduction where positive & negative charge carriers are accelerated in different directions due to $\vec{E}$ field

Interactions between different particle species, induce a damping force on electrons

e.g. Drude model

$$m \frac{d\langle \vec{v} \rangle}{dt} + m \frac{\langle \vec{v} \rangle}{\tau_R} = q\vec{E}$$

Simplest realization is in terms of Lorentz gas

Calculation of diffusion coefficient & heat conduction

We again consider two component system

e.g. Lorentz gas (interaction with static scattering centers)

radioactively marked particles in gas

Basic strategy same as usual

- Consider limit of Knudsen & Hilbert expansion
- Search for local equilibrium solution $f^{(0)}$

  (in this case with gradients in $T, \mu$)

- Construct non equilibrium correction $f^{(1)}$ according to first order of Hilbert expansion

$$\left(\frac{\partial}{\partial t} + \frac{\vec{p}}{m}\vec{\nabla}_{\vec{r}} + \vec{F}\vec{\nabla}_{\vec{p}}\right) f^{(0)} = C\left[f^{(1)}, f^{(0)}\right]$$

- Invert collision operator (e.g. in RTA)

- Calculate currents

$$\vec{J}_N = \int_{\vec{p}} \frac{\vec{p}}{m} f$$

$$\vec{J}_E = \int_{\vec{p}} \frac{\vec{p}}{m} \frac{\vec{p}^2}{2m} f$$

- read off coefficients in expansion of constitutive relations in affinities

e.g. $\vec{J}_N = \boxed{L_{NN}} \vec{\nabla}\left(-\frac{\mu}{T}\right) + \boxed{L_{NE}} \vec{\nabla}\left(\frac{1}{T}\right)$

Microscopically calculated coefficients automatically satisfy symmetry relations (Curie & Onsager principle)

# Hydrodynamics from Boltzmann equation

We have seen that the Boltzmann equation can describe relaxation of a non-equilibrium system towards global equilibrium based on microscopic description in terms of single particle phase space density

Eventually we expect that in the long time limit, evolution should become „simpler", i.e. it can be described by local thermodynamic variables as in Chapter I

Comparison:

BTE: $f(t,\vec{r},\vec{p})$ $\left(\frac{\partial}{\partial t} + \frac{\vec{p}}{m}\vec{\nabla}_r + \vec{F}\vec{\nabla}_p\right) f = C[f]$

Hydro: conservation laws for $e, n, \vec{p}$

$$\frac{\partial}{\partial t} n + \vec{\nabla} \vec{J}_N = 0 \quad \ldots$$

constitutive relations

$$J_N = J_N^{eq} + L_{N\alpha} F_\alpha$$

equation of state

$$n = n(T, \mu, \vec{v})$$

Clearly hydrodynamic description involves fewer dof's

Goal is to understand how reduction to small number of dof's is possible

We are interested in evolution of expectation values of one body operators, according to

$$\langle \chi(t,\vec{r},\vec{p}) \rangle_{\vec{p}} = \frac{1}{n(t,\vec{r})} \int_{\vec{p}} \chi(t,\vec{r},\vec{p})\, f(t,\vec{r},\vec{p})$$

and we know how to derive EOMs for such quantities from the Boltzmann equation

Specifically if $\chi(t,\vec{r},\vec{p})$ is a collisional invariant, we know that contributions from collision integral vanishes

$$\frac{\partial}{\partial t} n\langle \chi \rangle_{\vec{p}} - n\left\langle \frac{\partial \chi}{\partial t} \right\rangle_{\vec{p}} + n\int_{\vec{p}} \chi \frac{\vec{p}}{m} \vec{\nabla}_r f$$

$$+ n\int_{\vec{p}} \chi \vec{F} \vec{\nabla}_p f = n\langle \chi_f C[f] \rangle = 0$$

So upon integration by parts

$$\frac{\partial}{\partial t} n\langle \chi \rangle_{\vec{p}} - n\left\langle \frac{\partial \chi}{\partial t} \right\rangle_{\vec{p}} + \vec{\nabla}_r \left\langle n \chi \frac{\vec{p}}{m} \right\rangle_{\vec{p}} - n\left\langle \frac{\vec{p}}{m} \vec{\nabla}_r \chi \right\rangle_{\vec{p}}$$

$$- n\vec{F} \left\langle \vec{\nabla}_p \chi \right\rangle_{\vec{p}} = 0$$

Can now find equivalent form of balance equations convenient for derivation of fluid dynamic equations

Choosing $\chi = m$ $\quad \vec{\nabla}_r \chi = 0 \quad \vec{\nabla}_p \chi = 0 \quad \frac{\partial}{\partial t}\chi = 0$

$$\frac{\partial}{\partial t}\rho(t,\vec{r}) + \vec{\nabla}_r\left(\rho(t,\vec{r})\left\langle\frac{\vec{p}}{m}\right\rangle_{\vec{p}}\right) = 0$$

Defining $\boxed{\vec{V}(t,\vec{r}) = \left\langle\frac{\vec{p}}{m}\right\rangle_{\vec{p}}}$ we get the usual continuity eqn

$$\boxed{\frac{\partial}{\partial t}\rho(t,\vec{r}) + \vec{\nabla}_r\left(\rho(t,\vec{r})\,\vec{V}(t,\vec{r})\right) = 0}$$

Next we consider the kinetic momentum

$\chi^i = p^i \quad \vec{\nabla}_r \chi^i = 0 \quad \vec{\nabla}_{p_j}\chi^i = \delta^{ij} \quad \frac{\partial}{\partial t}\chi = 0$

$$\frac{\partial}{\partial t}\rho(t,\vec{r})\,v^i(t,\vec{r}) + \vec{\nabla}_r\left\langle\rho(t,\vec{r})\,\frac{p^i}{m}\frac{\vec{p}}{m}\right\rangle_{\vec{p}} = \frac{\rho(t,\vec{r})}{m}F^i(t,\vec{r})$$

where we encounter $\left\langle\frac{p^i}{m}\frac{p^j}{m}\right\rangle_{\vec{p}}$ as a new object. If the system features a nonzero velocity $\vec{V}(t,\vec{r})$ it makes sense to separate avg. values from result in local rest frame (lrf)

$$\left\langle\frac{\vec{p}_i}{m}\frac{\vec{p}_j}{m}\right\rangle_{\vec{p}} = \left\langle\left(\frac{\vec{p}_i}{m} - \vec{V}_i(t,\vec{r})\right)\left(\frac{\vec{p}_j}{m} - \vec{V}_j(t,\vec{r})\right)\right\rangle_{\vec{p}} + \vec{V}_i(t,\vec{r})\,\vec{V}_j(t,\vec{r})$$

Since the first term survives in lrf we can then identify

$$\boxed{\Pi^{ij}(t,\vec{r}) = \rho(t,\vec{r})\left\langle\left(\frac{p_i}{m} - V_i\right)\left(\frac{p_j}{m} - V_j\right)\right\rangle_{\vec{p}}} \quad \text{stress tensor}$$

So we are left with the term

$$n \left\langle \frac{p_j}{m} \frac{\partial v^i}{\partial r_j} \left( p^i - m v^i \right) \right\rangle_{\vec{p}}$$

$$= \underbrace{\underbrace{m n}_{\rho} \Bigg[ \left\langle \frac{p_i}{m} \frac{p_j}{m} \right\rangle - \underbrace{\left\langle v^i \frac{p_j}{m} \right\rangle}_{v^i v^j} \Bigg]}_{\equiv \Pi^{ij}} \frac{\partial v^i}{\partial r_j}$$

$$= \Pi^{ij} \frac{\partial v^i}{\partial r_j}$$

So collecting everything we find the balance equation

$$\boxed{\frac{\partial e(t,\vec{r})}{\partial t} + \vec{\nabla}\left( \vec{J}_u(t,\vec{r}) + e(t,\vec{r})\, \vec{v}(t,\vec{r}) \right) + \Pi_{ij} \frac{\partial v^i}{\partial r_j}(t,\vec{r}) = 0}$$

for the internal energy

we thus get the balance equation

$$\boxed{\frac{\partial}{\partial t}\rho(t,\vec{r})\,V^i(t,\vec{r}) + \frac{\partial}{\partial r_j}\left[\rho(t,\vec{r})\,V^i(t,\vec{r})\,V^j(t,\vec{r}) + \Pi^{ij}(t,\vec{r})\right] = \frac{1}{m}\rho(t,\vec{r})\,F^i(t,\vec{r})}$$

So far nothing really new but now let us consider energy. What we really care about a internal energy $e$ related to motion of gas particles in LRF, since change in velocity already described by momentum balance

$$\chi(t,\vec{r},\vec{p}) = \frac{\left(\vec{p}-m\vec{V}(t,\vec{r})\right)^2}{2m}$$

$$\frac{\partial}{\partial t}\chi = -\frac{\partial\vec{V}}{\partial t}\left(\vec{p}-m\vec{V}(t,\vec{r})\right)$$

$$\vec{\nabla}_{r_j}\chi = -\vec{\nabla}_{r_j}\vec{V}\cdot\left(\vec{p}-m\vec{V}(t,\vec{r})\right)$$

$$\vec{\nabla}_p\chi = \left(\frac{\vec{p}}{m}-\vec{V}(t,\vec{r})\right)$$

Define $\quad e(t,\vec{r}) = \frac{1}{2}\rho(t,\vec{r})\left\langle\left(\frac{\vec{p}}{m}-\vec{V}(t,\vec{r})\right)^2\right\rangle_{\vec{p}} = n\langle\chi\rangle_{\vec{p}}$

$$\frac{\partial}{\partial t}e(t,\vec{r}) + \rho\frac{\partial\vec{V}}{\partial t}\underbrace{\left\langle\frac{\vec{p}}{m}-\vec{V}\right\rangle_{\vec{p}}}_{=0} + \vec{\nabla}_k\left\langle n\frac{(\vec{p}-m\vec{V})^2}{2m}\frac{\vec{p}}{m}\right\rangle_{\vec{p}}$$
$$+\, n\left\langle\frac{p_j}{m}\frac{\partial V^i}{\partial r_j}\left(p^i-mV^i\right)\right\rangle_{\vec{p}}$$
$$+\, n\vec{F}\underbrace{\left\langle\frac{\vec{p}}{m}-\vec{V}\right\rangle_{\vec{p}}}_{=0} = 0$$

we recognize that

$$\left\langle n\frac{(\vec{p}-m\vec{V})^2}{2m}\frac{\vec{p}}{m}\right\rangle_{\vec{p}} = \underbrace{\left\langle n\frac{(\vec{p}-m\vec{V})^2}{2m}\left(\frac{\vec{p}}{m}-\vec{V}\right)\right\rangle}_{\text{energy that in LRF}} + \dots\vec{V}_i e$$

$$\vec{J}_u = \left\langle n\frac{(\vec{p}-m\vec{V})^2}{2m}\left(\frac{\vec{p}}{m}-\vec{V}\right)\right\rangle_{\vec{p}} = \frac{1}{2}\rho\left\langle\left(\frac{\vec{p}}{m}-\vec{V}\right)^2\left(\frac{\vec{p}}{m}-\vec{V}\right)\right\rangle_{\vec{p}}$$

Now this concludes part 1) of the derivation of hydrodynamics, what remains to be seen is how constitutive relations emerge from Boltzmann framework

Generally this will result in reduction of dof's to local thermodynamic properties and e.g. their gradients

→ Necessary to truncate Boltzmann equation

Note that different truncations are possible corresponding to different expansion schemes / power countings

→ active topic of research in particular for relativistic theories

We will discuss traditional way based on Chapman-Enskog expansion

which can be viewed as a short expansion in Knudsen number