Recap. Discussed solutions to BTE

global equilibrium solution:

Stationary, homogeneous solution in absence of external forces $f(t,\vec{r},\vec{p}) = f_{eq}(\vec{p}) = n\left(\frac{2\pi\hbar^2}{mk_BT}\right)^{3/2} \exp\left(-\frac{\vec{p}^2}{2mk_BT}\right)$

Scales in BTE

$C[f] \sim \frac{1}{\tau_{mfp}} f \qquad \tau_{mfp} \sim \frac{1}{n\sigma_{tot}\langle V\rangle}$

associated length scale $l_{mfp} \sim \langle V\rangle \tau_{mfp} \sim \frac{1}{n\sigma_{tot}}$

should be compared to other characteristic time/length scales

e.g. $\nabla_r f \sim \frac{1}{L} f$ or $\tau_{evol} \sim$ age of the system

Scale ratio defines Knudsen number

$Kn = \frac{l_{mfp}}{L}$

if $Kn \ll 1$ several collisions take place over the course of the response to external gradients

$\Rightarrow$ expect system to approach local equilibrium $f^{(0)}(t,\vec{r},\vec{p})$ which is a solution of $C[f^{(0)}] = 0$

We constructed local equilibrium solutions based on detailed balance property

$$f^{(0)}(t,\vec{r},\vec{p}) = n(t,\vec{r}) \left(\frac{2\pi\hbar^2}{mk_BT(t,\vec{r})}\right) \exp\left(-\frac{(\vec{p}-m\vec{V}(t,\vec{r}))^2}{2mk_BT(t,\vec{r})}\right)$$

Consider $Kn \ll 1$, can obtain approximate solution to Boltzmann dynamics by expanding around local equilibrium distribution

$$f(t,\vec{r},\vec{p}) = f^{(0)}(t,\vec{r},\vec{p}) + f^{(1)}(t,\vec{r},\vec{p}) + \ldots$$

where $f^{(0)}$ is local equilibrium distribution and $|f^{(1)}(t,\vec{r},\vec{p})|$ is a small correction

Dynamics of $f^{(1)}$ governed by linearized Boltzmann equation

→ Different strategies to solve depending on power-counting of residual terms in Boltzmann equation

Will discuss one particular expansion scheme which we will need to derive hydrodynamics from BTE

Generally we find that with such ansatz

$$\frac{\partial f^{(0)}}{\partial t} + \frac{\partial f^{(1)}}{\partial t} + \vec{v}\vec{\nabla}_r f^{(0)} + \vec{v}\vec{\nabla}_r f^{(1)}$$

$$+ \vec{F}\vec{\nabla}_p f^{(0)} + \vec{F}\vec{\nabla}_p f^{(1)} = C[f^{(0)} + f^{(1)}]$$

where

$$C[f^{(0)} + f^{(1)}] = C[f^{(0)}] + \delta C[f^{(0)}, f^{(1)}]$$

with

$$\delta C[f^{(0)}, f^{(1)}] = \frac{1}{2} \int_{\vec{p}_2}\int_{\vec{p}_3}\int_{\vec{p}_4} \left[f_3^{(0)} f_4^{(1)} + f_3^{(1)} f_4^{(0)} - f_1^{(0)} f_2^{(1)} - f_1^{(1)} f_2^{(0)}\right]$$
$$\tilde{W}(p_1 p_2 \to p_3 p_4)$$

Now if we power-count the individual terms what we will have

$$\delta C[f^{(0)}, f^{(1)}] \sim \frac{1}{\tau_{mfp}} f_1$$

$$\vec{v}\vec{\nabla}_r f_0 \sim \frac{1}{\tau_s} f_0 \sim K_n \frac{1}{\tau_{mfp}} f_0$$

Like case if we are interested in the approach to global equilibrium on time scales $\tau_{evol} \approx \tau_s$

$$\frac{\partial}{\partial t} f^{(0)} \sim \frac{1}{\tau_s} f_0 \sim K_n \frac{1}{\tau_{mfp}} f_0$$

Since on large time scales (~τ) deviations from local equilibrium (described by $f^{(1)}$) are in response to gradients of f_0, e.g. $\vec{v}\vec{\nabla}_r f^{(0)}$ we can find a consistent truncation by solving

$$\frac{|f^{(1)}|}{f^{(0)}} \sim K_n$$

~~thus we have~~

$O(K_n^0)$: $\quad C[f^{(0)}] = 0$

$\Rightarrow f^{(0)}$ is local equilibrium distribution

$O(K_n^1)$: $\quad \delta C[f^{(0)}, f^{(1)}] = \left(\frac{\partial}{\partial t} + \vec{v}\vec{\nabla}_r + \vec{F}\vec{\nabla}_p\right) f^{(0)}$

$\Rightarrow$ Solution for $f^{(1)}$ requires inversion of collision operator $\delta C[f^{(0)}, f^{(1)}]$

Hilbert expansion can be developed to higher orders
$\Rightarrow$ required for calculation of higher order transport coefficients

Discussion shows that solution to Boltzmann equation are hard to find even for small deviations from equilibrium

Often times for practical considerations one employs simplified collision kernels

Simplest possible version

Relaxation time approximation (RTA)

One assumes that small deviations from equilibrium will relax exponentially towards local equilibrium on a time scale $\tau_r(T(\vec{r}), n(\vec{r}), \vec{p})$

One then has

$$\delta C[f^{(0)}, f^{(1)}] = - \frac{f^{(1)}(t, \vec{r}, \vec{p})}{\tau_r(T(\vec{r}), n(\vec{r}), \vec{p})}$$

Even beyond linearized description RTA is frequently employed as approximation for full Boltzmann dynamics, in this case one approximates the full collision integral as

$$C[f] = - \frac{f - f^{(0)}}{\tau_r}$$

yielding

$$\underbrace{\left(\frac{\partial}{\partial t} + \vec{v}\vec{\nabla}_r + \vec{F}\vec{\nabla}_p\right) f = - \frac{f - f^{(0)}}{\tau_r}}$$

Boltzmann equation in relaxation time approximation

Note that in the latter case, the equilibrium distribution $f^{(0)}$ needs to be determined self-consistently

e.g. by Landau matching to determine

$\vec{V}(t,\vec{r})$, $T(t,\vec{r})$, $n(t,\vec{r})$

appearing in $f^{(0)}$

Since $f^{(0)}$ is characterized by 5 parameters need five additional equations to close non-linear RTA

e.g. $n[f] = \int d^3p\, f$ $\qquad$ $n[f] = n[f_0]$

$\vec{P}[f] = \int d^3p\, \vec{p} f$ $\qquad$ $\vec{P}[f] = \vec{P}[f_0]$

$e[f] = \int d^3p\, \frac{\vec{p}^2}{2m} f$ $\qquad$ $e[f] = e[f_0]$

Calculation of transport coefficients

What is a transport coefficient?

We encountered these in Chapter I as coefficients relating fluxes of conserved quantities $N, E, \vec{P}$ to affinities $\vec{\nabla}\left(\frac{1}{T}\right), \vec{\nabla}\left(-\frac{\mu}{T}\right), \vec{\nabla}\otimes\vec{v}$ which describe (small) deviations of the system from global equilibrium

$$\text{e.g.} \quad \vec{J}_N = \boxed{L_{NN}}\, \vec{\nabla}\left(-\frac{\mu}{T}\right)$$

Now for the treatment in Chapter I we treated system as close to local equilibrium

$\rightarrow$ justification requires $Kn \ll 1$ to maintain approx local equilibrium conditions

Can now employ Boltzmann machinery (??) to look at the dynamics from more microscopic perspective

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Electrical conductivity (DC)

Note that this is the DC conductivity, which
⇒ physically quite different from AC conductivity
at high frequency / wave number

Microscopic picture: Dilute gas of charged particles
subject to a static external electric field

$$\rightarrow \quad \vec{F}_L = q\vec{E}$$

and two-body collisions described by Boltzmann transport equation

⇒ collisions counteract acceleration of charged particles

Since $\vec{E}$ is uniform and static, consider expansion around homogeneous equilibrium state

$$f^{(0)}(t,\vec{r},\vec{p}) = f^{(0)}(\vec{p}) = n\left(\frac{2\pi\hbar^2}{mk_BT}\right) e^{-\frac{(\vec{p}-m\vec{v})^2}{2mk_BT}}$$

By proper choice of reference frame, we can eliminate $\boxed{\vec{v}=0}$ (local rest frame)*

* [illegible] that $|\vec{v}| \ll c$ is required as is not generally [illegible] E & M fields

Since we are interested in the response to electric field $\vec{E}$ over a time scale $\gg \tau_{mfp}$ can exploit Hilbert expansion to calculate induced change in phase space distribution due to $\vec{E}$-field

$$\delta C[f^{(0)}, f^{(1)}] = \left(\frac{\partial}{\partial t} + \vec{v}\vec{\nabla}_r + \vec{F}\vec{\nabla}_p\right) f^{(0)}$$

+ terms of higher order which will be neglected

Consider relaxation time approximation

$$\delta C[f^{(0)}, f^{(1)}] = -\frac{f^{(1)}(t, \vec{r}, \vec{p})}{\tau_R(T(\vec{r}), n(\vec{r}), \vec{p})}$$

- Since $\vec{\nabla}_r f^{(0)} = 0$ and $\frac{\partial}{\partial t} f^{(0)} = 0$ we have $T(\vec{r}) = \text{const}$ $n(\vec{r}) = \text{const}$

$$q\vec{E}\,\vec{\nabla}_p f^{(0)}(\vec{p}) = -\frac{f^{(1)}(t, \vec{r}, \vec{p})}{\tau_R(\vec{p})}$$

where

$$\vec{\nabla}_p f^{(0)}(\vec{p}) = \frac{-\vec{p}}{m k_B T} f^{(0)}(\vec{p})$$

So we obtain as stationary solution for $f^{(1)}$

$$f^{(1)}(\vec{r}, \vec{p}) = \frac{q}{m k_B T} \tau_R(\vec{p}) \, (\vec{p}\cdot\vec{E}) \, f^{(0)}(\vec{p})$$

Can now compute electron current

$$\vec{J}_{el}(t,\vec{r}) = \int_{\vec{p}} q\left(\frac{\vec{p}}{m}\right) f(t,\vec{r},\vec{p})$$

Since $f(t,\vec{r},\vec{p})$ was found to have no space-time dependence in this regime of interest, we get a uniform current

$$J^i_{el} = \frac{q}{m}\int_{\vec{p}} p^i\left[f^{(0)}(\vec{p}) + f^{(1)}(\vec{p})\right]$$

Now integral $p^i f^{(0)}(\vec{p})$ vanishes due to reflection symmetry

$$J^i_{el} = \underbrace{\frac{q^2}{m^2 k_B T}\int_{\vec{p}} \tau_R(\vec{p})\, p^i p^j f^{(0)}(\vec{p})}_{=\,\sigma^{ij}_{el}\ \text{electrical conductivity}}\, E^j$$

Specifically for $\tau_R(\vec{p}) = \tau_R = \text{const}$ we can use $\int d\Omega_p\, p^i p^j = \frac{1}{3}p^2\delta^{ij}$

$$J^i_{el} = \frac{2}{3}\frac{q^2\tau_R}{m k_B T}\underbrace{\left(\int d^3p\, \frac{p^2}{2m} f^{(0)}(\vec{p})\right)}_{\frac{3}{2} n k_B T}\delta^{ij} E^j$$

$$J^i_{el} = \underbrace{\left[\frac{q^2 n\tau_R}{m}\delta^{ij}\right]}_{=\,\sigma^{ij}_{el}} E^j$$

Note that Drude prediction came out automatically, so $\sigma^{ij} = \sigma\,\delta^{ij}$ as anticipated

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Note that physically we can interpret each factor in

$$\sigma_{el} = \frac{q^2 n \tau_{el}}{m}$$

q^2: charge per particle

n: number of particles

$\frac{1}{m}$: inertia of particle responding to $\vec{E}$

τ_{el}: time that particle can travel before encounter collision with $O(1)$ probability

Notably the Boltzmann RTA agrees with Drude model

$$m\frac{d\langle\vec{v}\rangle}{dt} + m\frac{\langle\vec{v}\rangle}{\tau_{el}} = q\vec{E}$$

$$V_{terminal} = \frac{q\vec{E}\tau_{el}}{m}$$

$$\vec{J} = qn V_{terminal} = \frac{q^2 n \tau_{el}}{m}\vec{E}$$

Hence the clear advantage is that Boltzmann formalism can be generalized to more complicated interactions