Recap:

Discussed balance equations for
$e, n, \vec{p}$

→ Since collisions do not change of total $e, n, \vec{p}$ based on microscopic conservation laws we obtain simple balance equations

e.g. particle number density

$$n(t,\vec{r}) = \int_{\vec{p}} f(t,\vec{r},\vec{p})$$

$$\frac{\partial}{\partial t} n + \vec{\nabla}_r \vec{J}_N = \int_{\vec{p}} C[f](t,\vec{r},\vec{p}) = 0$$

Next we discussed statistical entropy

→ Basic definitions for classical & quantum systems

→ Entropy is conserved under Hamiltonian dynamics

Boltzmann entropy:

$$S_B(t) = -k_B \int d^3r \int \frac{d^3p}{(2\pi\hbar)^3} f(t,\vec{r},\vec{p}) \log\left(f(t,\vec{r},\vec{p})\right)$$

We showed H-theorem

$$\frac{d}{dt} S_B(t) > 0$$

Loschmidt paradox: What causes irreversibility of BTE?

→ Negligence of N-body correlations induces irreversibility
Since correlations are effectively discarded after each collision → [illegible]

Solutions to Boltzmann equation

Generally solutions to non-linear integro differential equations are hard to find
→ even proof of existence & uniqueness only achieved in recent years for phenomenologically relevant interactions

We will discuss two kinds of solutions
- (near) equilibrium solutions
- (quasi) stationary non-equilibrium solutions

Equilibrium solutions:

Important information provided by H-theorem namely $\frac{dS}{dt} \geq 0$ implies that for stationary solutions $S(t)$ has to be maximal $\frac{dS}{dt} = 0$

Generally distinguish two types of solutions

global equilibrium: $\frac{\partial f_{eq}}{\partial t} = 0$

all terms in LHS and RHS of Boltzmann equation cancel and the system as a whole is in equilibrium
expect that this situation can be described by thermodynamic variables T, V, μ
spatially homogeneous in absence of external forces

Local equilibrium:

Denoted as $f^{(0)}$, where $f^{(0)}$ leads to a vanishing collision integral

$$C[f^{(0)}](t,\vec{r},\vec{p}) = 0 \quad \forall\ t,\vec{r},\vec{p}$$

Note that H-theorem does not guarantee convergence towards equilibrium in the limit $t \to \infty$

Counterexamples: 1) If phase space is unbounded system may continue expanding (e.g. heavy-ion collision)

2) Systems confined to a finite volume may show periodic behaviour

Consider spatially homogeneous system $f(t,\vec{r},\vec{p}) = f(t,\vec{p})$ independent of $\vec{r}$. In the absence of external forces, need to find solutions which lead to vanishing collision integral

$$\left(\frac{\partial f_1}{\partial t}\right)_{coll} = \int_{p_2}\int_{p_3}\int_{p_4} W(p_1p_2 \to p_3p_4)\left[f_3f_4 - f_2f_1\right] \stackrel{!}{=} 0$$

can distinguish two different situations

<u>balance</u>: $\int_{p_2}\int_{p_3}\int_{p_4} I_{coll} = 0$

<u>detailed balance</u>: $I_{coll} = 0 \quad \forall\ p_1, p_2, p_3, p_4$
satisfying energy-momentum conservation

global & local equilibrium solutions in the above sense satisfy <u>detailed balance</u>
$\Rightarrow$ each process cancelled by its inverse process

$$\boxed{f_1 f_2 = f_3 f_4}$$

now if $f > 0$ everywhere, we can take logarithm

$$\ln(f_1) + \ln(f_2) = \ln(f_3) + \ln(f_4)$$

$\Rightarrow \ln(f)$ is a conserved quantity in each microscopic scattering event

Can be expressed as linear combination of collisional invariants $\chi(\vec{p})$

Generally we then have

$$\ln(f(\vec{p})) = \sum_i \lambda_i \chi_i(\vec{p})$$

considering particle number energy & momentum conservation, we have

$$\ln(f(\vec{p})) = \lambda_N + \vec{\lambda}_{\vec{p}} \cdot \vec{p} + \lambda_E \frac{\vec{p}^2}{2m}$$

$$f_{eq}(\vec{p}) = e^{\lambda_N} \, e^{\vec{\lambda}_p \vec{p}} \, e^{\lambda_E \vec{p}^2}$$

where normalizability requires $\lambda_E < 0$

Solving $\boxed{n} = \int \frac{d^3 p}{(2\pi\hbar)^3} f_{eq}(\vec{p})$

and $E_{kin} = \int \frac{d^3 p}{(2\pi\hbar)^3} \frac{\vec{p}^2}{2m} f_{eq}(\vec{p}) = \frac{3}{2} N k_B \boxed{T}$

$$m n \boxed{\vec{v}_0} = \int \frac{d^3 p}{(2\pi\hbar)^3} \vec{p} \, f_{eq}(\vec{p})$$

we then find $n, T, \vec{v}$ in terms of $\lambda_N, \vec{\lambda}_p, \lambda_E$

yielding

$$\boxed{f_{eq}(\vec{p}) = n \left(\frac{2\pi\hbar^2}{m k_B T}\right)^{3/2} e^{-\frac{(\vec{p} - m\vec{v}_0)^2}{2m k_B T}}}$$

Maxwell-Boltzmann distribution

Note that $\vec{v}$ can be set to zero by appropriate frame choice (LRF) local equilibrium solutions:

Generally these are not solutions to the full Boltzmann equation

Only RHS vanishes but LHS does not

Nevertheless conceptually important because of hierarchy of time scales

fast

τ_c: time scale of single collisions
→ not actually described by Boltzmann equation

slow

τ_e, τ_s: System responds to gradients and external forces

Denoting the typical length scale of gradients
$\tau_e \sim \left(\vec{v} \frac{\vec{\nabla} f}{f} \right)^{-1} \sim \frac{L}{V_{th}}$
by a scale L

Should be compared to time scale for collisional relaxation which is a local process

$$\tau_{mfp} \sim \left(\frac{\left(\frac{\partial f}{\partial t}\right)_{coll}}{f} \right)^{-1} \sim \left(n\, \sigma_{tot}\, v_{th} \right)^{-1}$$

Using $v_{th} \tau_{mfp} \equiv l_{mfp}$ we find

$$l_{mfp} \sim \frac{1}{n\, \sigma_{tot}}$$

such that

$$\tau_{mfp} \sim \frac{l_{mfp}}{v_{th}}$$

So the relative importance of „local" collision events with global response to gradients is controlled by the ratio

$$\frac{\tau_{mfp}}{\tau_s} \sim \frac{l_{mfp}}{L} \equiv K_n \qquad \text{Knudsen number}$$

can find systematic solutions in the regime where

$$K_n \gg 1 \qquad \text{or} \qquad K_n \ll 1$$

free-streaming + corrections

many collisions occur before response to macroscopic gradients

Scales

$Kn \ll 1$ $\qquad\qquad\qquad$ $Kn \gg 1$

fast $\downarrow$ slow

τ_c $\qquad$ τ_c

τ_{mfp} $\qquad$ τ_s

τ_s $\qquad$ τ_{mfp}

where $\tau_c \ll \tau_{mfp}$ has to be satisfied always to ensure validity of approximations in Boltzmann equation

When $Kn \ll 1$ several collisions happen over the course of these two scales to respond to gradients

→ collisions lead to 'local' equilibrium of momentum distribution before system eventually relaxes towards global equilibrium

In this context it makes sense to characterize intermediate states by

<u>local equilibrium:</u>

$$f^{(0)}(t,\vec{r},\vec{p}) = n(t,\vec{r}) \left[\frac{2\pi\hbar^2}{mk_B T(t,\vec{r})}\right]^{3/2} e^{-\frac{(\vec{p}-m\vec{v}(t,\vec{r}))^2}{2mk_B T(t,\vec{r})}}$$

with local $n(t,\vec{r})$, $T(t,\vec{r})$, $\vec{v}(t,\vec{r})$