Recap: Heuristic derivation of Boltzmann equation

Starting point: short range interactions in a dilute gas ($r_0^3 \ll d^3$)

$\rightarrow$ mean free path $l_{mfp} \gg r_0$

Dynamics dominated by free-streaming with rare collision events

Coarse graining: Describe dynamics on time scales $\Delta t \gg t_{coll}$ and length scales $\Delta l \gg r_0$

$$\left(\frac{\partial}{\partial t} + \vec{v}\,\vec{\nabla}_r + \vec{F}(\vec{r})\,\vec{\nabla}_p\right) f_1(t,\vec{r},\vec{p}) = C[f](t,\vec{r},\vec{p})$$

Collision integral $C[f](t,\vec{r},\vec{p})$ describes local & instantaneous effects of collisions

molecular chaos: interacting particles uncorrelated before every collision

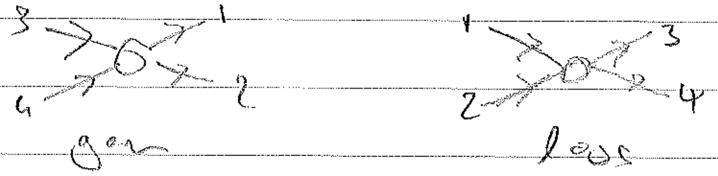
$$\frac{dN_{coll}}{dt} \sim f_1(t,\vec{r}_1,\vec{p}_1)\, f(t,\vec{r}_2,\vec{p}_2)\Big|_{\vec{r}_1 \approx \vec{r}_2}$$

Based on coarse graining, outcome of collisions only known on statistical basis

$$\tilde{W}(p_1 p_2 \rightarrow p_3 p_4)$$

Collision integral

$$C[f](t,\vec{r}_1,\vec{p}_1) = \frac{1}{2}\int \frac{d^3p_2}{(2\pi\hbar)^3}\int \frac{d^3p_3}{(2\pi\hbar)^3}\int \frac{d^3p_4}{(2\pi\hbar)^3}\; \tilde{W}(p_1 p_2 \to p_3 p_4)$$
$$\left[f(t,\vec{r},\vec{p}_3)\, f(t,\vec{r},\vec{p}_4) - f(t,\vec{r},\vec{p}_1)\, f(t,\vec{r},\vec{p}_2) \right]$$



Note $\frac{1}{2}$ pre-factor introduced for __identical particles__ (can also be absorbed into definition of $\tilde{W}$)

Comments: Boltzmann equation closed at one-body level

→ molecular chaos assumption / Stoßzahlansatz effectively truncates the BBGKY hierarchy

Coarse graining in space & time make the equation local in space and time

→ all single particle distributions evaluated at the same position $\vec{r}_1$ and time t

One often denotes

$$f_1(t,\vec{r}_1,\vec{p}_1) = f(t,\vec{r},\vec{p}) = f$$

$$f(t,\vec{r},\vec{p}_2) = f_2$$

$$f(t,\vec{r},\vec{p}_3) = f_3$$

$$\vdots$$

$$\left(\partial_t + \vec{v}\nabla_r + \vec{F}\nabla_p\right) f = -\frac{1}{2}\int_{p_2,p_3,p_4} \tilde{W}(p\,p_2 \to p_3 p_4)\left[f f_2 - f_3 f_4\right]$$

where $\int_{p_i} = \int \frac{d^3 p_i}{(2\pi\hbar)^3}$

Besides classical particles as considered in our example can also derive Boltzmann equation in quantum theory ($\rightsquigarrow$ QFT)

bosons: Bose enhancement

$$C[f] = -\frac{1}{2}\int_{p_2 p_3 p_4} \tilde{w}(p p_2 \to p_3 p_4)\left[f f_2 (1+f_3)(1+f_4) - f_3 f_4 (1+f)(1+f_2)\right]$$

fermions: Pauli blocking

$$C[f] = -\frac{1}{2}\int_{p_2 p_3 p_4} \tilde{w}(p p_2 \to p_3 p_4)\left[f f_2 (1-f_3)(1-f_4) - f_3 f_4 (1-f)(1-f_2)\right]$$

where classical particle limit is recovered for $f \ll 1$

Besides classical particles, the Boltzmann equation for bosons has an interesting limit $f \gg 1$ when only Bose enhanced processes are allowed

classical fields/waves:

$$C[f] = -\frac{1}{2}\int_{p_2 p_3 p_4} \tilde{w}(p p_2 \to p_3 p_4)\left[f f_2 (f_3 + f_4) - f_3 f_4 (f + f_2)\right]$$

which plays an important role e.g. in studying acoustic turbulence

Derivation of collision term from BBGKY hierarchy

We start from evolution equation for f_2

$$\left[\frac{\partial}{\partial t} + v_1 \vec{\nabla}_{r_1} + v_2 \vec{\nabla}_{r_2} + \vec{F}_1 \vec{\nabla}_{p_1} + \vec{F}_2 \vec{\nabla}_{p_2} + \vec{K}_{12}\left(\vec{\nabla}_{p_1} - \vec{\nabla}_{p_2}\right)\right] f_2$$

$$= - \int \left(K_{13} \nabla_{p_1} + K_{23} \nabla_{p_2}\right) f_3 \, d\Omega_3$$

Since in a dilute gas the RHS is suppressed by diluteness parameter

$$\frac{\int K_{12} \nabla_{p_1} f_3 \, d\Omega_3}{K_{12} \nabla_{p_1} f_2} \sim \frac{r_0^3}{d^3} \ll 1$$

this represents a higher order correction and will be neglected

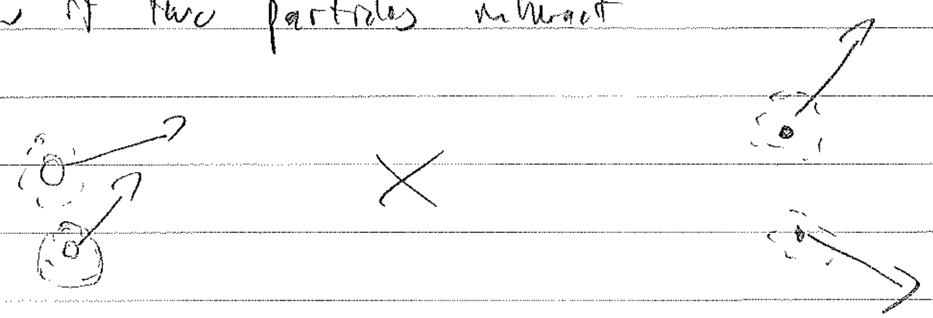
Note: On the level of heuristic derivation neglecting three body correlations of the arguments for molecular chaos

We have thus closed the BBGKY hierarchy at the two-body level

Next we consider the evolution of two body density,

$$\left[\frac{\partial}{\partial t} + \vec{v}_1 \vec{r}_1 + \vec{v}_2 \vec{r}_2 + \vec{F}_1 \vec{\nabla}_{p_1} + \vec{F}_2 \vec{\nabla}_{p_2} + \vec{K}_{12}\left(\vec{\nabla}_{p_1} - \vec{\nabla}_{p_2}\right)\right] f_2 = 0$$

Now if two particles interact



the COM momentum $\vec{p}_1+\vec{p}_2$ will not change whereas the relative momentum $\vec{p}_1-\vec{p}_2$ changes quite dramatically over the time scale of the collision.

Likewise, the COM coordinate $\vec{r}_1+\vec{r}_2$ changes slowly whereas the relative coordinate changes appreciably

Choice of coordinates

$$\vec{P} = \vec{p}_1 + \vec{p}_2 \qquad \Delta\vec{p} = \frac{1}{2}(\vec{p}_1 - \vec{p}_2)$$

$$\vec{R} = \frac{1}{2}(\vec{r}_1 + \vec{r}_2) \qquad \Delta\vec{r} = \vec{r}_2 - \vec{r}_1$$

$$\vec{\nabla}_{p_1} = \frac{\partial \vec{P}}{\partial \vec{p}_1}\vec{\nabla}_{\vec{P}} + \frac{\partial \Delta\vec{p}}{\partial \vec{p}_1}\vec{\nabla}_{\Delta\vec{p}} \equiv \vec{\nabla}_{\vec{P}} + \frac{1}{2}\vec{\nabla}_{\Delta\vec{p}}$$

$$\vec{\nabla}_{p_2} = \vec{\nabla}_{\vec{P}} - \frac{1}{2}\vec{\nabla}_{\Delta\vec{p}}$$

$$\vec{\nabla}_{r_1} = \frac{1}{2}\vec{\nabla}_R - \vec{\nabla}_{\Delta\vec{r}}$$

$$\vec{\nabla}_{r_2} = \frac{1}{2}\vec{\nabla}_R + \vec{\nabla}_{\Delta\vec{r}}$$

$$\left[\frac{\partial}{\partial t} + (\vec{F}_1 + \vec{F}_2)\vec{\nabla}_P + \frac{1}{2}(\vec{F}_1 - \vec{F}_2)\vec{\nabla}_{\Delta p} + \frac{1}{2}\frac{\vec{P}}{m}\vec{\nabla}_R - 2\frac{\Delta\vec{p}}{m}\vec{\nabla}_{\Delta\vec{r}} + K_{12}(|\Delta\vec{r}|)\vec{\nabla}_{\Delta\vec{p}}\right] f_2 = 0$$

Now since separation $|\Delta\vec{r}|$ has to be $\sim r_0$ (range of interaction) for collision to occur $\vec{r}_1 \simeq \vec{r}_2$ and therefore $\vec{F}_1 \simeq \vec{F}_2$ cancelling effects of external forces on relative motion

Since we consider $\Delta t \ll \tau_{ext}, \tau_s$ in coarse grained description effects of COM motion and forces will be negligible over this time scale

$\rightarrow$ can drop associated terms

So we have

$$\left[\frac{\partial}{\partial t} - 2\frac{\Delta\vec{p}}{m}\vec{\nabla}_{\Delta\vec{r}} + \vec{K}_{12}(|\Delta\vec{r}|)\vec{\nabla}_{\Delta\vec{p}}\right] f_2 \cong 0$$

Now we have $\frac{\partial}{\partial t}$ can be very large on time scales t_{coll}

However what we need for coarse grained description of one body distribution

$$\left.\frac{\partial f_1}{\partial t}\right|_{coll} = -\int d\Omega_2 \, \vec{K}_{12} \vec{\nabla}_{p_1} f_2$$

averaged over a time scale $\Delta t \gg t_{coll}$

Since rapid changes of f_2 do not contribute to long time average, can approximate f_2 by quasi-stationary solution $\left(\frac{\partial}{\partial t} f \simeq 0\right)$

$$\vec{K}_{12}(|\Delta\vec{r}|)\vec{\nabla}_{\Delta\vec{p}} f_2 = 2\frac{\Delta\vec{p}}{m}\vec{\nabla}_{\Delta\vec{r}} f_2$$

so we have

$$-\vec{K}_{12}(|\Delta\vec{r}|)\left(\vec{\nabla}_{p_1} - \vec{\nabla}_{p_2}\right) f_2 = -\left(\vec{V}_1 - \vec{V}_2\right)\vec{\nabla}_{\Delta\vec{r}} f_2$$

Now integrating over $\int d\Omega_2$ the $\vec{\nabla}_{p_2}$ contribution gives a vanishing boundary term

$$\left.\frac{\partial f_1}{\partial t}\right|_{coll} = \int d\Omega_2 (\vec{v}_2 - \vec{v}_1) \vec{\nabla}_{\Delta\vec{r}} f_2(t, \vec{r}_1, \vec{p}_1, \vec{r}_1 + \Delta\vec{r}, \vec{p}_2)$$

where $\int d\Omega_2 = \int \frac{d^3r_2\, d^3p_2}{(2\pi\hbar)^3} = \int \frac{d^3p_2}{(2\pi\hbar)^3} \int d^3\Delta r$

Now for each $\vec{p}_1, \vec{p}_2$ we can decompose $\Delta\vec{r}$ into components parallel and perpendicular to $(\vec{v}_2 - \vec{v}_1)$

$$\Delta\vec{r} = \frac{(\vec{v}_2 - \vec{v}_1)}{|\vec{v}_2 - \vec{v}_1|} z + \vec{b}$$

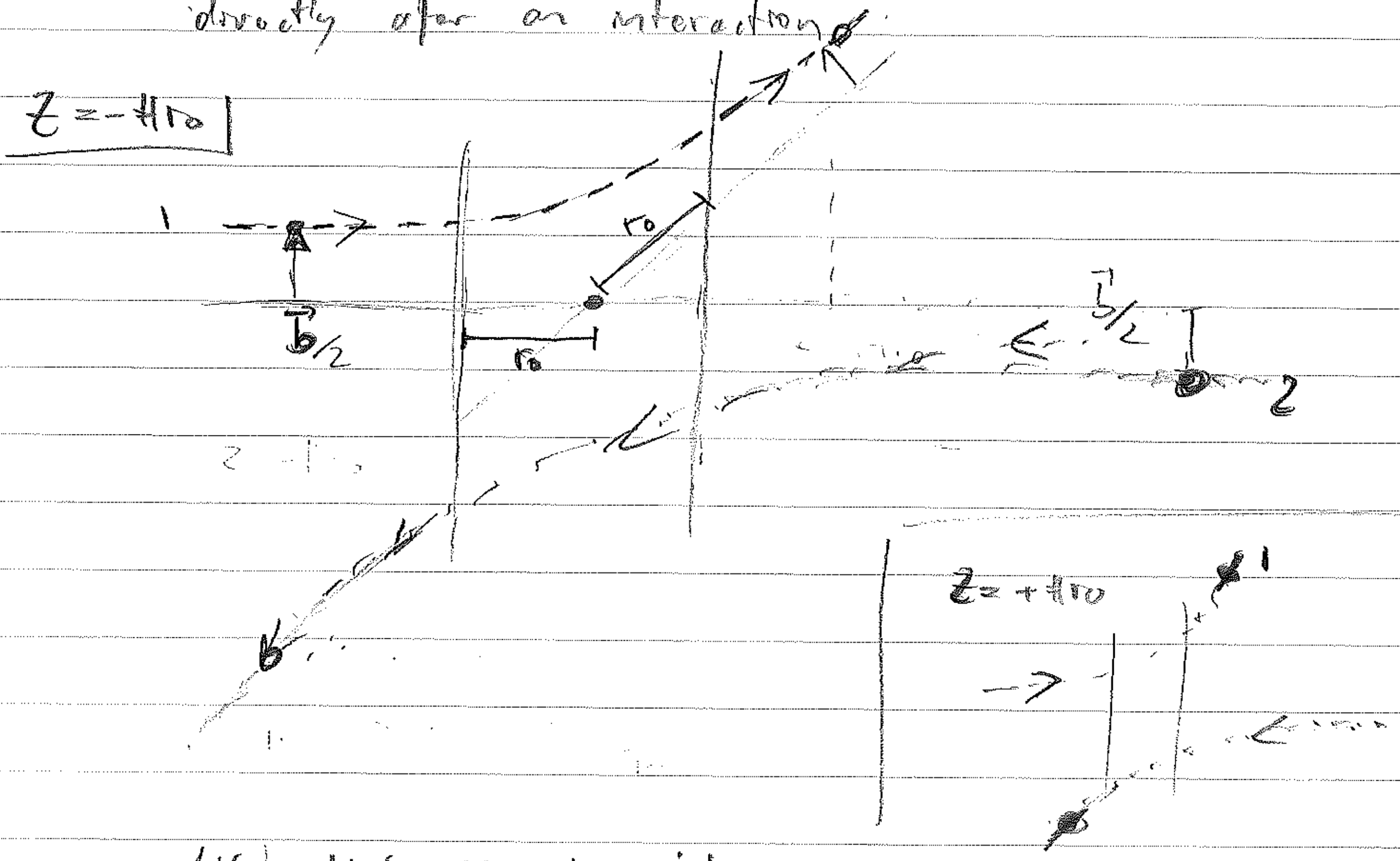
Then $(\vec{v}_2 - \vec{v}_1)\vec{\nabla}_{\Delta\vec{r}} = |\vec{v}_2 - \vec{v}_1| \frac{\partial}{\partial z}$

and $\int d^3\Delta r = \int dz \int d^2\vec{b}$

$$\left.\frac{\partial f_1}{\partial t}\right|_{coll} = \int \frac{d^2\vec{b}\, d^3p_2}{(2\pi\hbar)^3} |\vec{v}_2 - \vec{v}_1| \times \Big[f_2(t, \vec{r}_1, \vec{p}_1, \vec{r}_1 + \vec{b} + z\vec{e}_z, \vec{p}_2) \Big]_{z=-\#r_0}^{z=+\#r_0}$$

where # is a number of order unity

Now the key difference between upper and lower integration limit is that it describes ($z=-\#r_0$) a situation directly before an interaction or ($z=+\#r_0$) situation directly after an interaction.



Note that after interaction particles are correlated → non-trivial correlation, can not employ Stoßzahl ansatz

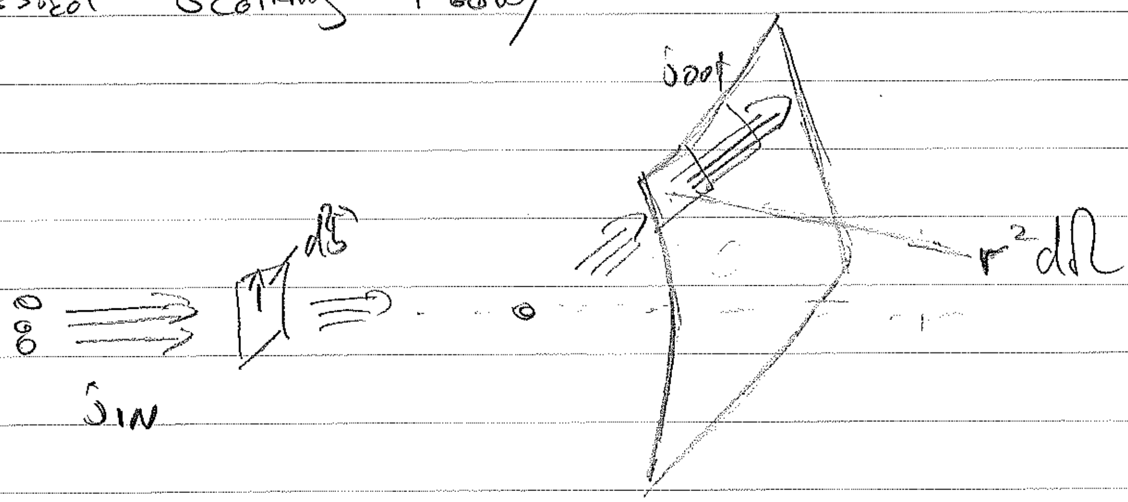
However since classical trajectories are deterministic can express probability to find 1,2 after collision terms of probability to find 3,4 before

$$f_2(t,\vec{r}_1,\vec{p}_1,\vec{r}_1+b+\#z_0\vec{e}_z,\vec{p}_2) = f_2(t,\vec{r}_1,\vec{p}_3,\vec{r}_1+b-\#z_0\vec{e}_z,\vec{p}_4)$$

where $p_{3/4}=p_{3/4}(b,\vec{p}_1,\vec{p}_2)$

Now to obtain Boltzmann equation in usual form still need to express $\int d^2b$ in terms of momentum integrals

Corresponding relation is given by classical scattering theory



$$\left|\frac{\vec{j}_{out}}{j_{IN}}\right|(E_{com}, \theta, \varphi) = \frac{1}{r^2}\frac{d\sigma}{d\Omega}$$

Now if we integrate over all solid angles at some distance the total fluxes have to match

$$\int |\vec{j}_{IN}|\, d^2\vec{b} = \int |j_{out}|\, r^2 d\Omega$$

$$= \int \frac{d\sigma}{d\Omega}\, d\Omega\, |j_{IN}|$$

where Ω is the scattering angle between the outgoing particles

So collecting everything

$$\frac{\partial f_1}{\partial t}\Big|_{coll} = \int \frac{d^3p_2}{(2\pi\hbar)^3} \int \frac{d\sigma}{d\Omega} d\Omega \; |\vec{v}_2 - \vec{v}_1| \Big[f_2(t, \vec{r}_1, \vec{p}_3, \vec{r}_1 + \vec{b} - \cancel{\#} r_0 \vec{e}_z, \vec{p}_4)$$

$$- f_2(t, \vec{r}_1, \vec{p}_1, \vec{r}_1 + \vec{b} - \# r_0 \vec{e}_z, \vec{p}_2) \Big]$$

Due to coarse graining we can neglect $\vec{b} - \# r_0 \vec{e}_z$ in the second argument of f_2 and employ the Stoßzahlansatz

$$f_2(t, \vec{r}_1, \vec{p}_3, \vec{r}_1, \vec{p}_4) = f_1(t, \vec{r}_1, \vec{p}_3) \, f_1(t, \vec{r}_1, \vec{p}_4)$$

etc.

to obtain the Boltzmann equation

We also learned that the transition rate is in fact given by

$$\frac{1}{2} \tilde{W}(p_1 p_2 \to p_3 p_4) \frac{d^3p_3}{(2\pi\hbar)^3} \frac{d^3p_4}{(2\pi\hbar)^3} = \frac{d\sigma}{d\Omega} d\Omega \, |\vec{v}_2 - \vec{v}_1|$$