Recap: Introduced concept of reduced distribution function

$$f_k(t, \vec{r}_1, \vec{p}_1, \dots \vec{r}_k, \vec{p}_k) = \int d^{6(N-k)}V \; f(t, \{\vec{r}_i\}, \{\vec{p}_i\})$$

We realized that to compute k-body observables we only need f_k

e.g. kin energy

$$\langle E_{kin} \rangle = \int \frac{d^3r_1 d^3p_1}{(2\pi\hbar)^3} \frac{p_1^2}{2m} f_1(t, \vec{r}_1, \vec{p}_1)$$

Motivated by this observation we derived evolution equation for k-body distribution

Considered system of neutral particles

$$H = \sum_{i=1}^{N} \frac{p_i^2}{2m} + \sum_{i=1}^{N} V(\vec{r}_i) + \sum_{\substack{i,j=1 \\ j>i}}^{N} W(|\vec{r}_i - \vec{r}_j|)$$

Starting from Liouville equation for full N-body distribution

$$\frac{d}{dt}f = \frac{\partial f}{\partial t} + \sum_{i=1}^{N} \frac{\vec{p}_i}{m} \vec{\nabla}_{r_i} f + \sum_{i=1}^{N} \vec{F}_i \vec{\nabla}_{p_i} f$$

$$+ \sum_{i=1}^{N} \sum_{\substack{j=1 \\ j\neq i}}^{N} K_{ij}(|\vec{r}_i - \vec{r}_j|) \vec{\nabla}_{p_i} f = 0$$

we integrated over $\int d^{6(N-k)}V$ to

obtain evolution equations for one and two body density

BBGKY Hierarchy

$$\frac{\partial f_1}{\partial t} + \vec{v}_1 \vec{\nabla}_{r_1} f_1 + \vec{F}(r_1) \vec{\nabla}_{p_1} f_1 = -\int d\Omega_2 \vec{K}_{12} \vec{\nabla}_{p_1} f_2$$

$$\left(\frac{\partial}{\partial t} + \vec{v}_1 \vec{\nabla}_{r_1} + \vec{v}_2 \vec{\nabla}_{r_2} + \vec{F}_1 \vec{\nabla}_{p_1} + \vec{F}_2 \vec{\nabla}_{p_2} + \vec{K}_{12} (\vec{\nabla}_{p_1} - \vec{\nabla}_{p_2}) \right) f_2$$

$$= -\int d\Omega_3 \left(\vec{K}_{13} \vec{\nabla}_{p_1} + \vec{K}_{23} \vec{\nabla}_{p_2} \right) f_3$$

. . . .

Generally we observe that evolution equation for N-body distribution involves N+1 body distribution

Coupled set of evolution equations which is only closed at N-body level, i.e. considering all particles

So far exact re-formulation of the problem, no approximations made except for limiting ourselves to two-body interactions

Generally the power of the formalism derives from the fact that under appropriate circumstances the hierarchy can be truncated by neglecting higher order correlations

Systems of charged particles

→ need to consider presence of vector potential

$$H = \sum_{i=1}^{N} \frac{(\vec{p}_i - q\vec{A}(t,\vec{r}_i))^2}{2m} + \sum_{i=1}^{N} q\phi(t,\vec{r}_i) + \sum_{\substack{i=1 \\ j>i}}^{N} W(|\vec{r}_i - \vec{r}_j|)$$

Now we have EOMs

$$\dot{\vec{r}}_i = \frac{\vec{p}_i - q\vec{A}(t,\vec{r}_i)}{m}$$ (no longer independent of position)

$$\dot{\vec{p}}_i = -q\vec{\nabla}_{r_i}\phi(t,\vec{r}_i) + \sum_{j\neq i} \vec{K}_{r_j} + q(\dot{\vec{r}}_i \cdot \vec{\nabla}_{r_i})\vec{A}(t,\vec{r}_i) + q\dot{\vec{r}}_i \times [\vec{\nabla}_{r_i} \times \vec{A}(t,\vec{r}_i)]$$

(no longer independent of momentum)

Changes steps in derivation of BBGKY Hierarchy

Can show that if coordinates rotated linear / kinetic momentum

$$p_{i,kin} = m\dot{\vec{r}}_i$$

same hierarchy can be derived (see Sec 4B,C in Pottier

$$\left(\frac{\partial}{\partial t} + \vec{v}_1 \vec{\nabla}_{r_1} + \vec{F}_L \vec{\nabla}_{p_1^{kin}}\right) f(t, \vec{r}_1, \vec{p}_1^{kin})$$

$$= -\int \frac{d^3 r_2 \, d^3 p_2^{kin}}{(2\pi\hbar)^3} \vec{K}_{12} \vec{\nabla}_{p_1^{kin}} f_2(t, \vec{r}_1, \vec{p}_1^{kin}, \vec{r}_2, \vec{p}_2^{kin})$$

where $\vec{F}_L$ is the Lorentz force

$$\vec{F}_L = q\left(\vec{E}_{ext} + \frac{\vec{p}_{kin}}{m} \times \vec{B}_{ext}\right)$$

due to external fields

Now one reason that this is extremely ... is that electromagnetic forces are long-range, for non-relativistic moving particles $\frac{v}{c} \ll 1$ dominated by static Coulomb interaction

$$W(|r_i - r_j|) = -\frac{q^2}{4\pi\varepsilon_0} \frac{1}{|\vec{r}_i - \vec{r}_j|}$$

$$\vec{K}_{ij} = \frac{q^2}{4\pi\varepsilon_0} \frac{\vec{r}_i - \vec{r}_j}{|\vec{r}_i - \vec{r}_j|^3}$$

unless there are screening phenomena present can not use short range approximation even for dilute media

Since force exerted is due to many particles, which can be very far away there is self-averaging which allows to approximate

$$f_2(t, r_1, p_1, r_2, p_2) \simeq f_1(t, r_1, p_1)\, f_2(t, r_2, p_2) + \underbrace{\text{correlations}}_{\text{neglected}}$$

<u>Vlasov / Mean-field approximation</u>

Mean field: Closed set of equations at one body level

$$\left[\frac{\partial}{\partial t} + \vec{v}_1 \vec{\nabla}_r + q\left(\vec{E}_{ext} + \vec{E}_{int} + \vec{v}\times\vec{B}_{ext}\right)\vec{\nabla}_{p_1}\right] f_1(t_1, \vec{r}_1, \vec{p}_1) = 0$$

where now one also has to account for internal Coulomb field

$$\vec{E}_{int}(t_1 \vec{r}_1) = \frac{q^2}{4\pi\varepsilon_0}\int \frac{d^3r_2\, d^3p_2}{(2\pi\hbar)^3}\, \frac{\vec{r}_1 - \vec{r}_2}{|\vec{r}_1 - \vec{r}_2|^3}\, f_1(t, \vec{r}_2, \vec{p}_2)$$

Note that this equation is non-linear and therefore non-trivial. It must be solved in self-consistent way.

Since we have only considered static Coulomb exchange, there are no internal $\vec{B}$ fields created due to currents in the plasma (suppressed by $\frac{v}{c}$)

More generally one has to consider single particle evolution equation along with Maxwell equations to describe $\vec{E}, \vec{B}$ fields including retardation effects

Scales in BBGKY

In order to derive meaningful approximations one has to identify the relevant scales

First look at system of non-interacting particles

Single particle Liouville equation / collisionless Boltzmann equation

$$\frac{\partial f_1(t,\vec{r},\vec{p})}{\partial t} + \vec{v}\,\vec{\nabla}_r f_1(t,\vec{r},\vec{p}) + \vec{F}(t,\vec{r})\,\vec{\nabla}_p f_1(t,\vec{r},\vec{p}) = 0$$

Now we see that there are two distinct time scales related to

$$\tau_s \sim (\vec{v}\,\vec{\nabla}_r)^{-1} \quad \text{and} \quad \tau_e \sim (\vec{F}\,\vec{\nabla}_p)^{-1}$$

describes the time scale on which spatial gradients of the distribution relax

describes the time scale on which momentum distribution responds to external forces

Near equilibrium one has

$$v \sim \sqrt{\frac{k_B T}{m}} \sim \frac{k_B T}{\sqrt{m k_B T}}$$

$$\vec{F} = -\vec{\nabla}_r V$$

$$\vec{\nabla}_p f \sim \nabla_p e^{-\frac{p^2}{2mk_BT}}$$

$\nabla_{\vec{r}} f$

$$\sim \frac{p}{mk_BT} f \sim \frac{f}{\sqrt{mk_BT}}$$

So

$$\frac{1}{\tau_s} \sim \vec{v}(\vec{\nabla}_r f) \sim \frac{k_B T}{\sqrt{m k_B T}} \nabla_r f$$

$$1/\tau_e \sim (\nabla_r V) \frac{f}{\sqrt{m k_B T}}$$

typically thermal energy $k_B T \gtrsim V$ (except for ultra-cold gases)
and
length scales of potential vary more slowly than length scales of phase space distribution $\frac{\nabla_r f}{f} \gtrsim \frac{\nabla_r V}{V}$

So $1/\tau_s \gtrsim \frac{1}{\tau_e} \Rightarrow \tau_e \gtrsim \tau_s$

meaning that a lot of microscopic dynamics happens during response to external influence

Next consider the effects of two-body interactions

$\rightarrow$ now two scales associated with interactions

$$\tau_c \sim \left(\vec{K}\,\vec{\nabla}_p\right)^{-1}$$

Now typically when two particles interact

$$\vec{K} \approx \vec{\nabla} W$$

where W change in potential energy can be comparable to thermal energy $k_B T$

$$W \sim k_B T$$

However typically the interactions are short range
i.e. interaction potential varies over very short (microscopic) distance scale

$$\frac{\nabla_r W}{W} \gg \frac{\nabla_r f}{f}$$

Hence

$$1/\tau_c \sim \frac{\nabla_r W}{W} k_B T \frac{1}{\sqrt{m k_B T}}$$

$$1/\tau_s \sim \frac{\nabla_r f}{f} k_B T \frac{1}{\sqrt{m k_B T}}$$

So $1/\tau_c \gg 1/\tau_s \Rightarrow \tau_c \ll \tau_s$

collision time much smaller than spatial relaxation rates

$\rightarrow$ many collisions happen during the time that system responds to spatial gradients

Now there is one interesting difference between evolution equation of f_1 and f_2

$$\left(\frac{\partial}{\partial t} + \vec{v}\vec{\nabla}_r + \vec{F}\vec{\nabla}_p\right) f_1 = -\int_{\Omega_2} \vec{K}_{12}\vec{\nabla}_{p_1} f_2$$

$$\left(\frac{\partial}{\partial t} + \vec{v}_1\vec{\nabla}_{r_1} + \vec{v}_2\vec{\nabla}_{r_2} + \vec{F}_1\vec{\nabla}_{p_1} + \vec{F}_2\vec{\nabla}_{p_2} + \vec{K}_{12}\left(\vec{\nabla}_{p_1} - \vec{\nabla}_{p_2}\right)\right) f_2$$

$$= -\int_{\Omega_3} \left(K_{13}\vec{\nabla}_{p_1} + K_{23}\vec{\nabla}_{p_2}\right) f_3$$

Now if we consider a dilute system
we have a small probability to find many
particles in a small phase space volume, so