Inflation

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Overview

Definitions

$\hbar = c = k_B = \varepsilon_0 = 1$: natural units $\rightarrow$

$m_{\text{Pl}} = 1/\sqrt{8\pi G}$: reduced Planck mass

$R_{\mu\nu}$: Ricci tensor, R : Ricci scalar

$a(t)$: scale factor

$H := \frac{\dot{a}}{a}$: Hubble parameter

H^{-1} : Hubble time

$\frac{a(t_{\text{end}})}{a(t_{\text{init}})}$: expansion factor of the universe between times t_{init} and t_{end}

N : number of e-foldings between times t_{init} and t_{end} $\left(e^N = \frac{a(t_{\text{end}})}{a(t_{\text{init}})}\right)$

$l_{H,t} := a(t) \int_{t_{\text{Pl}}}^{t} \frac{dt}{a(t)}$: cosmological hoirzon size at time $t \gg t_{\text{Pl}}$

Hot Big Bang theory

Standard Cosmology is based on the observation that the universe is homogeneous, isotropic, flat and warm on large scales.
Described by the Friedmann–Lemaître–Robertson–Walker metric

$$ds^2 = dt^2 - a(t)^2 \left(\frac{dr^2}{1 - Kr^2} + r^2 d\theta^2 + r^2 \sin^2 \theta d\varphi^2 \right),$$

with t the cosmic time, $a(t)$ the Robertson-Walker scale factor, and the three possibilities $K > 0, K = 0$ and $K < 0$ being known as a closed, flat and open space-time respectively.
This however, leads to some problems.[1]

[1] Kaloian Lozanov: Lectures on Reheating after Inflation 4.1

Flatness problem

From the Einstein equation

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = \frac{T_{\mu\nu}}{m_{\mathsf{Pl}}^2} + g_{\mu\nu}\Lambda,$$

one can find two equations governing the evolution of the scale factor a, known as the Friedmann and Raychaudhuri equations respectively. To derive them, one uses the facts that, due to homogeneity, $T_{\mu\nu}$ only depends on t, near the origin of locally Cartesian, co-moving coordinates, and furthermore that, due to the isotropy, $T^{i0} = 0$ and $T^{ij} \propto \delta^{ij}$ hold. One also sets $\Lambda = 0$ as it is known to be negligibly small in the early universe.

Flatness problem

Then, by defining the energy density ρ and the pressure p of the perfect fluid filling the universe, through $T^{00} = \rho(t)$ and $T^{ij} = -a(t)^{-2}\delta^{ij}p(t)$ respectively, one obtains the following forms for the Friedmann equation

$$\left(\frac{\dot{a}}{a}\right)^2 \equiv H^2 = \frac{\rho}{3m_{\mathsf{Pl}}^2} - \frac{K}{a^2},$$

and the Raychaudhuri equation

$$\frac{\ddot{a}}{a} = -\frac{\rho + 3p}{6m_{\mathsf{Pl}}^2}.$$

Flatness problem

On the other hand, the energy-conservation ($T^{0\mu}{}_{;\mu}$), which, in this case, reads as

$$\dot{\rho} + 3H(\rho + p) = 0,$$

combined with a constant EOS ($w = p/\rho$), implies $\rho \propto a^{-3-3w}$. Insertion into the FR eq's leads to a solution $a(t) \propto t^{2/(3+3w)}$. Thus for both dust ($w = 0$), as well as radiation ($w = 1/3$), leading to $a \propto t^{2/3}$ and $a \propto t^{1/2}$, respectively, $\ddot{a} < 0$ holds, i.e. the universe always decelerates.
By defining $\Omega := \rho/\rho_c$, with the critical energy density $\rho_c := 3m_{\mathrm{Pl}^2}H^2$, and $\Omega_K := -K/(a^2H^2)$, one can rewrite the Friedmann equation as

$$\Omega + \Omega_K = 1.$$

Flatness problem

From the most recent observations of the anisotropies in the CMB, we know $|\Omega_{K,0}| < 0.005$. However, since $\ddot{a} < 0$, $\dot{a} = aH$ increases when going backwards in time, this value must be lower, when looking at earlier times. In other words as $t \to 0$, $\Omega \to 1$ with unnaturally high precision. For example, at the time of Big Bang Nucleosynthesis $|\Omega - 1| \ll 10^{-17}$, and during the GUT epoch $|\Omega - 1| \ll 10^{-55}$.

This constitutes the flatness problem, as a slight deviation from the necessary initial condition $\rho(t = 0) = \rho_c$ would lead to the universe either expanding too rapidly, or collapsing, before the large scale structures that exist today, could form.[2]

[2] Kaloian Lozanov: Lectures on Reheating after Inflation 4.1

Horizon problem

Another problem arises, when considering the causal structure of the universe under the premise of deceleration. The physical length scale, defined as $l_{\text{phys}} := al$, with l a constant, co-moving length scale, increases as the universe expands. However, it's ratio with the Hubble radius H^{-1} decreases, since $l_{\text{phys}}/H^{-1} = \dot{a}l$ and $\ddot{a} < 0$. This means that at early times, any co-moving length scale exceeds the Hubble scale, which in turn implies, that the causally-connected region in the universe today should lie deep inside the Hubble volume.

Horizon problem

Quantitatively, we consider the size of a causally connected region at the time of recombination t_{rec}, given by the cosmological horizon

$$l_{H,\text{rec}} = a(t_{\text{rec}}) \int_{t_{\text{Pl}}}^{t_{\text{rec}}} \frac{dt}{a(t)},$$

which, by now, has been stretched to

$$l_{H,\text{rec}}(t_0) = a(t_0) \int_{t_{\text{Pl}}}^{t_{\text{rec}}} \frac{dt}{a(t)},$$

due to the expansion of the universe.

Horizon problem

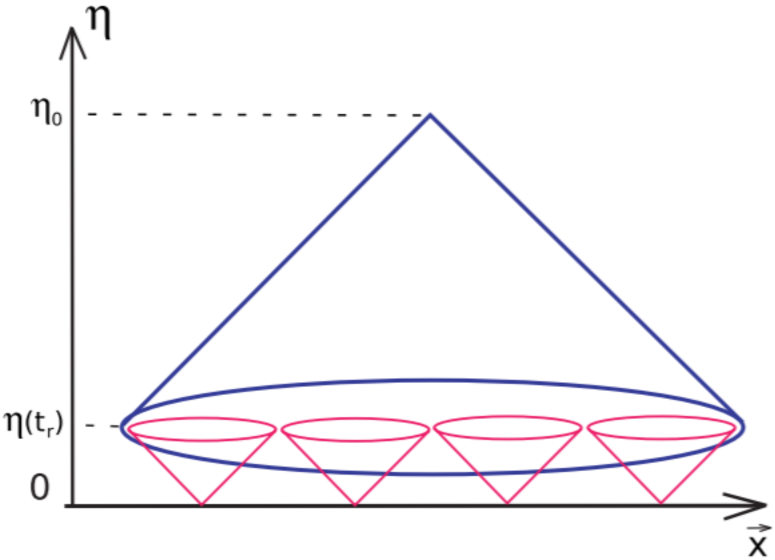
The current size of a region that was causally connected at the time of recombination is then simply $(l_{H,\mathrm{rec}}(t_0))^2$. Thus, by estimating the ratio between $l_{H,\mathrm{rec}}(t_0)$ and the cosmological horizon size today

$$\frac{l_{H,0}}{l_{H,\mathrm{rec}}(t_0)} = \frac{\int_0^{t_0} \frac{dt}{a(t)}}{\int_0^{t_{\mathrm{rec}}} \frac{dt}{a(t)}} \simeq \frac{a(t_{\mathrm{rec}}H(t_{\mathrm{rec}})}{a(t_0)H(t_0)} = \sqrt{1+z(t_{\mathrm{rec}})},$$

we can find the number of causally disconnected regions in the sphere of last scattering observed via the CMB:

$$\left(\frac{l_{H,0}}{l_{H,\mathrm{rec}}(t_0)}\right)^2 \simeq z(t_{\mathrm{rec}}) \simeq 1000.$$

Horizon problem



Horizon problem

This is, what is known as the horizon problem, since the CMB is very homogeneous, which would be near impossible if these ~ 1000 predicted regions had never been in causal contact.[3]

[3] D.S.Gorbunov, V.A.Rubakov: "Introduction to the theory of the early universe: Cosmological perturbations and inflationary theory" 11.1.1

The inflaton field

The most common model of inflation, consistent with observations, is that of a single scalar field ϕ, introduced via the action

$$S = \int d^4x \sqrt{-g} \left[-\frac{m_{\mathsf{Pl}}^2}{2} R + \frac{1}{2} \nabla_\mu \phi \nabla^\mu \phi - V(\phi) \right] + S_{\mathsf{matter}}.$$

Here S_{matter} contains the standard model action, as well as all the other information about the matter sector, including the terms coupling the ϕ to other fields. The field ϕ is called the inflaton, and is the single source, of the accelerated expansion in this model of inflation.[4]

[4] Kaloian Lozanov: Lectures on Reheating after Inflation 4.2

Homogeneous dynamics

Due to homogeneity and isotropy, the dominant component of the inflaton field $\bar{\phi}$ only depends on t. One can also find the following expressions for the energy density and pressure of the $\bar{\phi}$ - fluid

$$\rho_{\bar{\phi}} = \frac{1}{2}\dot{\bar{\phi}}^2 + V(\bar{\phi}), \quad p_{\bar{\phi}} = \frac{1}{2}\dot{\bar{\phi}}^2 - V(\bar{\phi}).$$

Recalling the Raychaudhuri equation

$$\frac{\ddot{a}}{a} = -\frac{\rho + 3p}{6m_{\mathsf{Pl}}^2},$$

one finds the condition for accelerated expansion is $\rho_{\bar{\phi}} + 3p_{\bar{\phi}} < 0$, or equivalently $\dot{\bar{\phi}}^2 < V(\bar{\phi})$, after inserting the formulae above.

Homogeneous dynamics

We can write down the Friedmann equation

$$H^2 = \frac{1}{3m_{\mathrm{Pl}}^2}\left[\frac{1}{2}\dot{\bar{\phi}}^2 + V(\phi)\right],$$

and the Euler-Lagrange equation

$$\ddot{\bar{\phi}} + 3H\dot{\bar{\phi}} + \partial_{\bar{\phi}} V(\bar{\phi}) = 0,$$

for $\bar{\phi}$.[5]

[5] Kaloian Lozanov: Lectures on Reheating after Inflation 4.2.1

Slow-roll inflation

As we will show later, for inflation to solve the problems of the HBBT, it needs to be long enough to achieve up to 100 e-folds. For accelerated expansion we need $\ddot{a} = a(\dot{H} + H^2) > 0$, which is equivalent to $\epsilon_H := -\dot{H}/H^2 < 1$. It is intuitive, that for inflation to last long, we need both $\epsilon_H \ll 1$ and and the same for it's rate of change, quantified by

$$\eta_H := \frac{\dot{\epsilon_H}}{H\epsilon_H} = \frac{d\ln\epsilon_H}{dN},$$

where we have used $dN = d\ln a = Hdt$.

Slow-roll inflation

Using the Friedmann and Raychaudhuri equations we find

$$\begin{aligned}\epsilon_H &= -\frac{\dot{H}}{H^2} = -\frac{\ddot{a}}{a}\frac{1}{H^2} + 1 \\ &= -\left(-\frac{\rho_{\bar{\phi}} + 3p_{\bar{\phi}}}{6m_{\text{Pl}}^2}\right)\left(\frac{3m_{\text{Pl}}^2}{\rho_{\bar{\phi}}}\right) + 1 \\ &= \frac{3}{2}\left(1 + \frac{p_{\bar{\phi}}}{\rho_{\bar{\phi}}}\right) = \frac{3}{2}\left(\frac{\rho_{\bar{\phi}} + p_{\bar{\phi}}}{\rho_{\bar{\phi}}}\right) = \frac{3}{2}\frac{\dot{\bar{\phi}}^2}{\rho_{\bar{\phi}}},\end{aligned}$$

which means $\epsilon_H \ll 1$ is equivalent to $\dot{\bar{\phi}}^2/2 \ll V(\bar{\phi})$. Thus $V(\bar{\phi})$ needs to be very flat, such that $\bar{\phi}$ rolls slowly.

Slow-roll inflation

For this so called slow-roll inflation to last long enough, to achieve the sufficient number of e-folds, we need the change of $\dot{\bar{\phi}}$ during one expansion time H^{-1} to be small, i.e. $|\ddot{\bar{\phi}}/\dot{\bar{\phi}}|H^{-1} \ll 1$. Inserting this into the Euler-Lagrange equation, as well as $\epsilon_H \ll 1$ into the Firedmann equation, leads to $\epsilon_H \simeq \epsilon_V$ and $\epsilon_H - |\ddot{\bar{\phi}}/\dot{\bar{\phi}}|H^{-1} \simeq \eta_V$, with

$$\epsilon_V := \frac{m_{\mathsf{Pl}}^2}{2}\left(\frac{\partial_{\bar{\phi}} V(\bar{\phi})}{V}\right)^2, \quad \eta_V := m_{\mathsf{Pl}}^2 \frac{\partial_{\bar{\phi}}^2 V(\bar{\phi})}{V},$$

the so called potential slow-roll parameters. It can be shown *, that $\epsilon_V \ll 1$ implies $N \gg \frac{\bar{\phi}(t_{\mathsf{Pl}})-\bar{\phi}(t_e)}{m_{\mathsf{Pl}}} \equiv \frac{\Delta\bar{\phi}}{m_{\mathsf{Pl}}}$, meaning a change in the value of the inflaton field of the order of the Planck mass would suffice, to give rise to a large number of e-foldings, during the accelerated expansion.

Slow-roll inflation

In general, models with a potential that supports slow-roll inflation for $\Delta\bar{\phi} \sim m_{\text{Pl}}$ are called chaotic inflation models. Basically all monomial potentials are in this class. Let us, in particular look at the simplest case $V(\bar{\phi}) = m^2\bar{\phi}^2/2$, where an approximate analytic solution, known as the attractor solution, can be found:

$$\dot{\bar{\phi}} \simeq -\sqrt{\frac{2}{3}} m_{\text{Pl}} m, \quad a \simeq a(t_{\text{Pl}}) \exp\left[\frac{mt}{\sqrt{6 m_{\text{Pl}}}}\left(\bar{\phi}(t_{\text{Pl}}) - \frac{m_{\text{Pl}} m}{\sqrt{6}} t\right)\right],$$

which breaks down towards the end of inflation $\bar{\phi} \simeq m_{\text{Pl}}$.[6]

[6] Kaloian Lozanov: Lectures on Reheating after Inflation 4.2.2

Transition to reheating

Towards the end of the slow-roll inflation the inflaton begins to oscillate around its potential minimum. Using the Friedmann and Euler-Lagrange equations, we can find the attractor solution oscillating around a quadratic minimum, which reads as

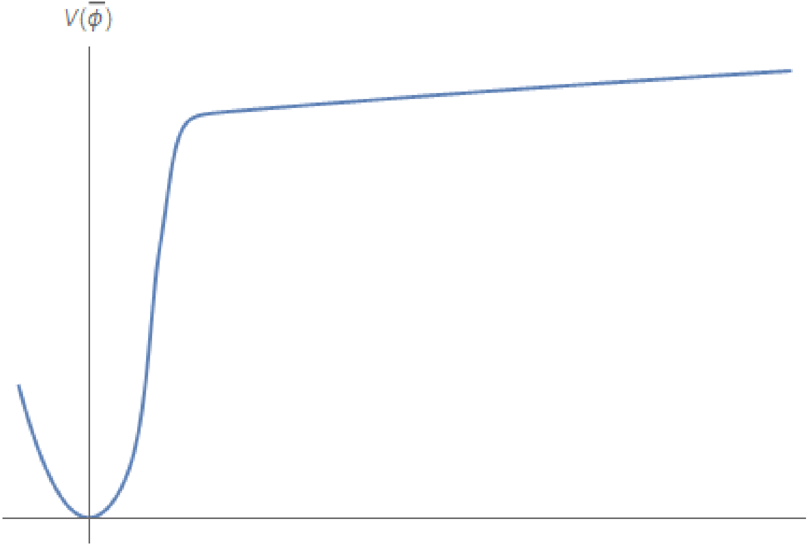
$$H \simeq \frac{2}{3t}\left(1+\frac{\sin(2mt)}{2mt}\right), \quad \bar{\phi} \simeq \sqrt{\frac{8}{3}}\frac{m_{\mathsf{Pl}}\cos(mt)}{mt}\left(1+\frac{\sin(2mt)}{2mt}\right).$$

One can also show, that a quadratic potential leads to a matter-like EOS ($w=0$), and a quartic one to a radiation-like EOS ($w=1/3$), however, potentials leading to an inflationary EOS ($w=-1/3$), would have a singular first derivative at their minimum, which is physically inconsistent. Thus, the oscillation around the potential minimum always leads to decelerated expansion.[7]

[7] Kaloian Lozanov: Lectures on Reheating after Inflation 4.2.3

Transition to reheating

A possible inflation potential could look like this



Solving the flatness and horizon problem

Recall the flatness problem, i.e. the curvature term Ω_K needing to be very small at early times, which makes it an unnatural initial condition. To prevent small changes from changing the outcome too drastically, we need a value closer to 1, i.e. $\Omega_K(t_{\mathsf{Pl}})/\Omega_K(t_0) \lesssim 1$. This can be recast as

$$1 \gtrsim \frac{[a(t_{\mathsf{Pl}})H(t_{\mathsf{Pl}})]^2}{(a_0 H_0)^2} = \frac{[a(t_{\mathsf{Pl}})H(t_{\mathsf{Pl}})]^2}{[a(t_e)H(t_e)]^2} \frac{[a(t_e)H(t_e)]^2}{(a_0 H_0)^2},$$

where t_e is the time where inflation ends.

Solving the flatness and horizon problem

This condition also solves the horizon problem, as, the current size of a region, which was causally connected at time t_e, is

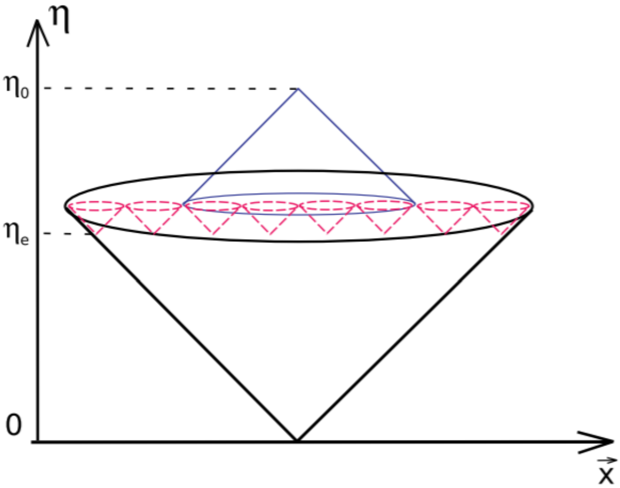
$$l_{H,e}(t_0) = a_0 \int_{a(t_{\text{Pl}})}^{a(t_e)} \frac{dt}{a(t)} = a_0 \int_{t_{\text{Pl}}}^{t_e} \frac{da}{Ha^2} \simeq \frac{a_0}{a(t_{\text{Pl}})H(t_{\text{Pl}})},$$

which holds due to the accelerated expansion, making the last integral be saturated at its lower bound. Thus

$$\frac{l_{H,e}(t_0)}{l_{H,0}} \simeq \frac{a_0 H_0}{a(t_{\text{Pl}})H(t_{\text{Pl}})} \gtrsim 1,$$

holds, i.e. the observable universe was causally connected at time t_e, if the estimate above holds.

Solving the flatness and horizon problem



Solving the flatness and horizon problem

Whether that estimate is fulfilled depends on the duration of inflation, or in other words, on the number of e-foldings that occur during it. For a rough estimate, we can assume reheating to be instantaneous, which means $a(t_e)/a_0 \sim T_0/T_{\text{reh}}$. We can then rewrite the necessary estimate as

$$\frac{T_0}{T_{\text{reh}}} \frac{H(t_{\text{Pl}})}{H_0} \lesssim \frac{a(t_e)}{a(t_{\text{Pl}})} \equiv e^{N_e},$$

where N_e is the number of e-foldings from the Plack-epoch till the end of inflation. For the estimate to be fulfilled we need

$$N_e > N_e^{(\text{min})} \simeq \log\left(\frac{T_0}{H_0}\right) \log\left(\frac{M_{\text{Pl}}}{T_{\text{reh}}}\right) \simeq 70 - 100,$$

for $T_{\text{reh}} = M_{\text{Pl}} - 1TeV$. (equiv. to $\Delta t_{\text{infl}}^{(\text{min})} \sim 10^{-42} - 10^{-9}s$)[8]

[8]D.S.Gorbunov, V.A.Rubakov: "Introduction to the theory of the early universe: Cosmological perturbations and inflationary theory" 11.2

Solving the entropy problem

The issue of the entropy problem is, that for the currently observed entropy of the universe, there would have to be an unnatural initial condition for the entropy. Inflation solves this, as any realistic model of inflation comes with a (p)reheating mechanism, which is capable of producing large amounts of entropy. Thus there is no need for the entropy to be large initially, which solves the problem.[9]

[9] D.S.Gorbunov, V.A.Rubakov: "Introduction to the theory of the early universe: Cosmological perturbations and inflationary theory" 11.2

Solving the primordial perturbation problem

Lastly, the primordial perturbation problem is also solved by inflation. The idea is, that microscopical density differences, cause by quantum fluctuations, were stretched to cosmological sizes during the inflationary period. It can be shown that predictions for these fluctuations, from explicit models of inflation, fit the observations of the CMB very well.[10]

[10] D.S.Gorbunov, V.A.Rubakov: "Introduction to the theory of the early universe: Cosmological perturbations and inflationary theory" 11.2

The End

Exponentially large expansion

$$e^N \equiv \frac{a(t_e)}{a(t_{\mathsf{Pl}})} = \exp\left[\int_{t_{\mathsf{Pl}}}^{t_e} H(t)dt\right]$$
$$= \exp\left[\int_{\bar{\phi}(t_{\mathsf{Pl}})}^{\bar{\phi}(t_e)} H(\bar{\phi})\frac{d\bar{\phi}}{\dot{\bar{\phi}}}\right] \simeq \exp\left[-m_{\mathsf{Pl}}^2 \int_{\bar{\phi}(t_{\mathsf{Pl}})}^{\bar{\phi}(t_e)} \frac{V(\bar{\phi})}{\partial_{\bar{\phi}} V(\bar{\phi})} d\bar{\phi}\right].$$

Assume $0 < V(\bar{\phi}(t_e)) < V(\bar{\phi}(t_{\mathsf{Pl}}))$ and $\bar{\phi}(t_{\mathsf{Pl}}) - \bar{\phi}(t_e) > 0$, then $\epsilon_V \ll 1$ implies $N \gg \frac{\bar{\phi}(t_{\mathsf{Pl}}) - \bar{\phi}(t_e)}{m_{\mathsf{Pl}}}$.